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Generalized Mangler-type principal value integrals with an application to fracture mechanics

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Abstract: A generalized interpretation of the classical Mangler-type principal value integrals is proposed. This new interpretation in general is not equivalent to the classical one. Several formulae for this new class of integrals are proved, concerning their relationship to Cauchy-type, generalized Cauchy-type and ordinary Mangler-type principal value integrals. Moreover, the Plemelj formulae and quadrature rules are considered for the same class of integrals. Finally, the present results are applied to the hypersingular integrodifferential equation of an interface crack in the classical theory of plane elasticity, which takes now the form of a simple hypersingular integral equation.

Keywords: Cauchy-type integrals, cracks, elasticity, finite-part integrals, fracture mechanics, hypersingular integrals, integrodifferential equations, interface cracks, Mangler-type integrals, numerical integration rules, Plemelj formulae, principal value integrals, second-order singularities, singular integrals.

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1. Introduction

In a recent note [6] we considered the application of the classical Mangler-type principal value integrals, so popular in fluid dynamics [1], to crack problems in fracture mechanics, where they have been implicitly used several times since 1982. (Few related references can be found in [6].) These integrals are strongly related to derivatives of Cauchy-type singular integrals, as well as to finite-part hypersingular integrals with a second-order singularity [6].

On the other hand, in [4] we have generalized the principal value interpretation of Cauchy-type integrals, having also provided several formulae for the resulting generalized Cauchy-type principal value integrals and their numerical evaluation, as well as an application to the interface crack problem in fracture mechanics. For this particular problem, following the rather recent, but completely justified from physical reasons, tendency to use the crack opening displacement along the crack instead of its first derivative in simple crack problems, we have recently reduced it to a hypersingular integrodifferential equation [5].

In this paper, we will generalize the fundamental results of [4] for Cauchy-type integrals to the equally important case of Mangler-type integrals. We will prove the related fundamental formulae, the numerical integration formulae and, finally, the simplification of the aforementioned hypersingular integrodifferential equation of the interface crack problem [5]. We hope that this new class of

hypersingular principal value integrals will prove useful in future both in fracture mechanics and in related problems of the theory of elasticity, as well as in other analogous problems of applied mathematics.

2. Definition

We reconsider the classical definition of Mangler-type principal value integrals [1, p. 133]

$$\oint_a^b \frac{f(t)}{(t-x)^2} dt = \lim_{\varepsilon \rightarrow 0} \left[\int_a^{x-\varepsilon} \frac{f(t)}{(t-x)^2} dt + \int_{x+\varepsilon}^b \frac{f(t)}{(t-x)^2} dt - \frac{2f(x)}{\varepsilon} \right], \quad a < x < b, \quad \varepsilon > 0, \quad (1)$$

where the integrand $f(x)$ is assumed to possess a continuous second derivative.

Using an approach analogous to that used in [4] for the simpler class of Cauchy-type principal value integrals, we define the generalized Mangler-type principal value integrals by permitting ε in (1) to take two different positive values in the two integrals of the right side of (1), that is,

$$\begin{aligned} {}^{(c)}\oint_a^b \frac{f(t)}{(t-x)^2} dt &= \lim_{\substack{\varepsilon' \rightarrow 0 \\ \varepsilon'' \rightarrow 0}} \left[\int_a^{x-\varepsilon'} \frac{f(t)}{(t-x)^2} dt + \int_{x+\varepsilon''}^b \frac{f(t)}{(t-x)^2} dt - \left(\frac{1}{\varepsilon'} + \frac{1}{\varepsilon''} \right) f(x) \right], \\ a < x < b, \quad \varepsilon', \varepsilon'' > 0, \end{aligned} \quad (2)$$

where the values of ε' and ε'' are restricted by

$$\lim_{\substack{\varepsilon' \rightarrow 0 \\ \varepsilon'' \rightarrow 0}} \frac{\varepsilon'}{\varepsilon''} = k \quad \text{with} \quad k = e^c, \quad k > 0, \quad (3)$$

the symbol c appearing in front of the integral sign in (2) to denote the proposed generalized definition of Mangler-type principal value integrals.

3. Properties

We will prove the following fundamental result:

Theorem 1: *Generalized Mangler-type principal value integrals are related to the corresponding ordinary integrals by*

$${}^{(c)}\oint_a^b \frac{f(t)}{(t-x)^2} dt = \oint_a^b \frac{f(t)}{(t-x)^2} dt + c f'(x), \quad f \in C^2, \quad a < x < b. \quad (4)$$

Proof: In the following, we assume continually that $a < x < b$. Since $f \in C^2$, it is obviously convenient to take into account the first three terms of the Taylor series of $f(t)$ in the neighbourhood of $t = x$:

$$f(t) = f(x) + (t-x)f'(x) + (1/2)(t-x)^2 f''(x) + O[(t-x)^3], \quad (5)$$

rewriting the ordinary integrals in the right side of the definition (2) as

$$\int_a^{x-\varepsilon'} \frac{f(t)}{(t-x)^2} dt = \int_a^{x-\varepsilon'} \frac{f(t) - f(x) - (t-x)f'(x)}{(t-x)^2} dt + f(x) \int_a^{x-\varepsilon'} \frac{dt}{(t-x)^2} + f'(x) \int_a^{x-\varepsilon'} \frac{dt}{t-x}, \quad (6a)$$

$$\int_{x+\varepsilon''}^b \frac{f(t)}{(t-x)^2} dt = \int_{x+\varepsilon''}^b \frac{f(t) - f(x) - (t-x)f'(x)}{(t-x)^2} dt + f(x) \int_{x+\varepsilon''}^b \frac{dt}{(t-x)^2} + f'(x) \int_{x+\varepsilon''}^b \frac{dt}{t-x}. \quad (6b)$$

Moreover, we have (under the restriction (3))

$$\lim_{\substack{\varepsilon' \rightarrow 0 \\ \varepsilon'' \rightarrow 0}} \left[\int_a^{x-\varepsilon'} \frac{dt}{t-x} + \int_{x+\varepsilon''}^b \frac{dt}{t-x} \right] = \ln \frac{b-x}{x-a} + \lim_{\substack{\varepsilon' \rightarrow 0 \\ \varepsilon'' \rightarrow 0}} \left[\ln \frac{\varepsilon'}{\varepsilon''} \right] = \ln \frac{b-x}{x-a} + \ln k = \ln \frac{b-x}{x-a} + c \quad (7)$$

because of (3). Since

$$\int_a^b \frac{dt}{t-x} = \ln \frac{b-x}{x-a}, \quad (8)$$

we can, in passing, rewrite (7) as

$$\lim_{\substack{\varepsilon' \rightarrow 0 \\ \varepsilon'' \rightarrow 0}} \left[\int_a^{x-\varepsilon'} \frac{dt}{t-x} + \int_{x+\varepsilon''}^b \frac{dt}{t-x} \right] = \int_a^b \frac{dt}{t-x} + c = {}^{(c)}\int_a^b \frac{dt}{t-x} \quad (9)$$

in accordance with the definition of generalized Cauchy-type principal value integrals suggested in [4]. In an analogous way, we find directly that

$$\int_a^{x-\varepsilon'} \frac{dt}{(t-x)^2} + \int_{x+\varepsilon''}^b \frac{dt}{(t-x)^2} = -\left(\frac{1}{x-a} + \frac{1}{b-x} \right) + \frac{1}{\varepsilon'} + \frac{1}{\varepsilon''}. \quad (10)$$

Therefore, because of (7) and (10), we obtain by adding (6) and using the formula of the definition (2) of generalized Mangler-type principal value integrals (of course, under the restriction (3))

$${}^{(c)}\int_a^b \frac{f(t)}{(t-x)^2} dt = \int_a^b \frac{f(t) - f(x) - (t-x)f'(x)}{(t-x)^2} dt - \left(\frac{1}{x-a} + \frac{1}{b-x} \right) f(x) + \left(\ln \frac{b-x}{x-a} + c \right) f'(x). \quad (11)$$

This formula permits us to evaluate generalized Mangler-type principal value integrals by using ordinary nonsingular integrals (existing under the restriction, already posed, that $f(x)$ possesses a continuous second derivative as is clear from (5)). Furthermore, the derivation of (11) makes also clear why the nonintegral terms in the right side of the definition (2) had to be used for the existence of the corresponding limiting value ((10) taken also into consideration).

Finally, since we can use (11) in the special case of ordinary Mangler-type principal value integrals (that is, with $k = 1$ and, therefore, $c = 0$), whence

$$\int_a^b \frac{f(t)}{(t-x)^2} dt = \int_a^b \frac{f(t) - f(x) - (t-x)f'(x)}{(t-x)^2} dt - \left(\frac{1}{x-a} + \frac{1}{b-x} \right) f(x) + \ln \frac{b-x}{x-a} f'(x), \quad (12)$$

we conclude directly (by comparing (11) and (12)) the validity of (4). $\square$

Evidently, beyond (4), we feel that (11) is also a fundamental formula since it relates generalized Mangler-type principal value integrals to regular integrals.

4. The Plemelj formulae

It is well known that Mangler-type principal value integrals are also derivatives of the corresponding Cauchy-type principal value integrals [6], that is,

$$\int_a^b \frac{f(t)}{(t-x)^2} dt = \frac{d}{dx} \left[\int_a^b \frac{f(t)}{t-x} dt \right]. \quad (13)$$

On the other hand, the classical Plemelj formulae (reported also in [4] and, in a generalized form, in [2]) hold for the limiting values of the sectionally analytic function

$$F(z) = \int_a^b \frac{f(t)}{t-z} dt, \quad z = x + iy \notin [a, b], \quad (14)$$

that is,

$$F^\pm(x) = \pm \pi i f(x) + \int_a^b \frac{f(t)}{t-x} dt. \quad (15)$$

By differentiating this formula and using (13), we obtain

$$F'^\pm(x) = \pm \pi i f'(x) + \int_a^b \frac{f(t)}{(t-x)^2} dt. \quad (16)$$

Finally, because of the fundamental property (4) of generalized Mangler-type principal value integrals, (16) can be modified as follows

$$F'^{\pm}(x) = \pm(\pi i \mp c)f'(x) + {}^{(c)}\oint_a^b \frac{f(t)}{(t-x)^2} dt. \quad (17)$$

These are the Plemelj formulae with generalized Mangler-type principal value integrals. Obviously, the following formula results also directly from (17)

$$\left(1 + \frac{c}{\pi i}\right)F'^+(x) + \left(1 - \frac{c}{\pi i}\right)F'^-(x) = 2 {}^{(c)}\oint_a^b \frac{f(t)}{(t-x)^2} dt. \quad (18)$$

Both (17) and (18) are generalizations of the corresponding results for generalized Cauchy-type principal value integrals [4]. Moreover, (18) is of particular interest during the solution of boundary value problems for the derivatives of sectionally analytic functions. Such an application will be made in Section 7.

5. Quadrature rules

We consider any classical interpolatory quadrature rule for ordinary integrals with n nodes t_{in} (and corresponding weight coefficients A_{in}) of the form

$$\int_a^b w(t)g(t) dt = \sum_{i=1}^n A_{in}g(t_{in}) + E_n, \quad (19)$$

where $w(t)$ denotes the weight function and E_n the corresponding error term. For Cauchy-type principal value integrals and their derivatives (equivalent, because of (13), to ordinary Mangler-type principal value integrals), we have (retaining, as usual, the symbol E_n for the error term) [3]

$$\oint_a^b w(t) \frac{g(t)}{t-x} dt = \sum_{i=1}^n A_{in} \frac{g(t_{in})}{t_{in}-x} + \frac{q_n(x)}{p_n(x)} g(x) + E_n, \quad x \neq t_{in}, \quad i = 1, \dots, n, \quad (20)$$

$$\oint_a^b w(t) \frac{g(t)}{(t-x)^2} dt = \sum_{i=1}^n A_{in} \frac{g(t_{in})}{(t_{in}-x)^2} + \frac{d}{dx} \left[\frac{q_n(x)}{p_n(x)} \right] g(x) + \frac{q_n(x)}{p_n(x)} g'(x) + E_n, \quad x \neq t_{in}, \quad i = 1, \dots, n. \quad (21)$$

The new functions $p_n(x)$ and $q_n(x)$ in these formulae are (as usual) defined by [3]

$$p_n(x) = \prod_{i=1}^n (x - t_{in}), \quad (22)$$

$$q_n(x) = \oint_a^b w(t) \frac{p_n(t)}{t-x} dt, \quad (23)$$

whence, because of (13),

$$q'_n(x) = \oint_a^b \frac{p_n(t)}{(t-x)^2} dt. \quad (24)$$

We also notice in passing that (21) can be interpreted as resulting through a formal differentiation of (20). In any case, the uniform convergence of both (20) and (21) was studied in detail in [3] under appropriate, but sufficiently general, conditions. The fundamental result (4) is applicable to the quadrature rules derived in this section as well.

Now, because of (4), we generalize (21) to the case of generalized Mangler-type principal value integrals. We find directly that

$$\begin{aligned} {}^{(c)}\oint_a^b w(t) \frac{g(t)}{(t-x)^2} dt &= \sum_{i=1}^n A_{in} \frac{g(t_{in})}{(t_{in}-x)^2} + \frac{d}{dx} \left[\frac{q_n(x)}{p_n(x)} \right] g(x) \\ &\quad + \frac{q_n(x)}{p_n(x)} g'(x) + c \frac{d}{dx} [w(x)g(x)] + E_n, \quad x \neq t_{in}, \quad i = 1, \dots, n, \end{aligned} \quad (25)$$

since $f(x)$ is equal to $w(x)g(x)$ in our case. The above equation can also be written as

$$\begin{aligned} {}^{(c)}\int_a^b w(t) \frac{g(t)}{(t-x)^2} dt &= \sum_{i=1}^n A_{in} \frac{g(t_{in})}{(t_{in}-x)^2} + \frac{d}{dx} \left[\frac{q_n(x)}{p_n(x)} + cw(x) \right] g(x) \\ &+ \left[\frac{q_n(x)}{p_n(x)} + cw(x) \right] g'(x) + E_n, \quad x \neq t_{in}, \quad i = 1, \dots, n. \end{aligned} \quad (26)$$

But since (because of (23))

$$q_n^{(c)}(x) \equiv {}^{(c)}\int_a^b \frac{p_n(t)}{t-x} dt = q_n(x) + cw(x)p_n(x) \quad (27)$$

because of the definition of generalized Cauchy-type principal value integrals [4], we can rewrite (26) as

$$\begin{aligned} {}^{(c)}\int_a^b w(t) \frac{g(t)}{(t-x)^2} dt &= \sum_{i=1}^n A_{in} \frac{g(t_{in})}{(t_{in}-x)^2} + \frac{d}{dx} \left[\frac{q_n^{(c)}(x)}{p_n(x)} \right] g(x) \\ &+ \left[\frac{q_n^{(c)}(x)}{p_n(x)} \right] g'(x) + E_n, \quad x \neq t_{in}, \quad i = 1, \dots, n. \end{aligned} \quad (28)$$

This is our fundamental formula for the numerical evaluation of generalized Mangler-type principal value integrals. It is quite similar to (21) for the corresponding ordinary integrals reducing also to it for $c = 0$. Moreover, (28) can be considered resulting by a formal differentiation of the corresponding quadrature rule for generalized Cauchy-type principal value integrals [4].

Finally, because of the construction of (28), it is obvious that the convergence results of [3] hold also true for the present quadrature rule as well without modifications.

6. Relation to generalized Cauchy-type integrals

It is well known (and already reported in (13)) that ordinary Mangler-type principal value integrals can be considered as the derivatives of the corresponding Cauchy-type integrals. The same holds also true for the corresponding generalized integrals (with the same value of c) that is,

$${}^{(c)}\int_a^b \frac{f(t)}{(t-x)^2} dt = \frac{d}{dx} \left[{}^{(c)}\int_a^b \frac{f(t)}{t-x} dt \right]. \quad (29)$$

This is a fundamental property of generalized Mangler-type principal value integrals. For its proof it is sufficient to use the formula [4]

$${}^{(c)}\int_a^b \frac{f(t)}{t-x} dt = \int_a^b \frac{f(t)}{t-x} dt + cf(x) \quad (30)$$

for generalized Cauchy-type principal value integrals and differentiate it, taking simultaneously into account the corresponding formula (4) for the generalized Mangler-type principal value integrals together with the differentiation formula (13).

The formulae of this section and of the previous two sections show clearly the strong relationship between generalized Mangler-type and Cauchy-type principal value integrals both in formal differentiation and in applications (like the Plemelj formulae and numerical integration rules). Now, finally, we will illustrate the application of generalized Mangler-type principal value integrals to a concrete problem from the mathematical theory of elasticity (fracture mechanics).

7. The interface crack

We consider a crack $[a, b]$ along the interface (for convenience coinciding with the Ox -axis) of two perfectly bonded isotropic elastic half-planes with shear moduli and Muskhelishvili's constants μ_1, κ_1 (for the upper half-plane) and μ_2, κ_2 (for the lower half-plane), respectively. This problem was recently reduced [5] to the following hypersingular integrodifferential equation

$$Af'(x) + \frac{B}{\pi i} \oint_a^b \frac{f(t)}{(t-x)^2} dt + \int_a^b k(t, x)f(t) dt = p(x), \quad a < x < b, \quad (31)$$

where $f(x)$ is an unknown function proportional to the crack opening displacement, $p(x)$ is the known loading distribution on both edges of the crack $[a, b]$ and A and B are appropriate bimaterial constants, defined by [5]

$$A = \mu_1 \mu_2 \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{(\kappa_1 \mu_2 + \mu_1)(\kappa_2 \mu_1 + \mu_2)}, \quad (32a)$$

$$B = \mu_1 \mu_2 \frac{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)}{(\kappa_1 \mu_2 + \mu_1)(\kappa_2 \mu_1 + \mu_2)}. \quad (32b)$$

Here, on the basis of the fundamental result (4), we can rewrite (31) in the following simpler form

$$\frac{B}{\pi i} {}^{(c)}\oint_a^b \frac{f(t)}{(t-x)^2} dt + \int_a^b k(t, x)f(t) dt = p(x), \quad a < x < b, \quad (33)$$

where, obviously,

$$c = \pi i \frac{A}{B}. \quad (34)$$

The quadrature rules of Section 5 can be easily used for the approximation of the integrals in (33) and its reduction to a system of linear algebraic equations approximating (33).

Furthermore, we notice that the crack problem under consideration (with $k(t, x) \equiv 0$) reduces to the following generalized (with derivatives of boundary values of a sectionally analytic function) Riemann–Hilbert boundary value problem on the crack $[a, b]$ [5]

$$\frac{\mu_1 \mu_2}{\pi i} \left[\frac{F'^+(x)}{\kappa_1 \mu_2 + \mu_1} + \frac{F'^-(x)}{\kappa_2 \mu_1 + \mu_2} \right] = p(x), \quad a < x < b, \quad (35)$$

where the sectionally analytic function $F(z)$ is a Cauchy-type integral of the form (14) with density the previously defined function $f(x)$. Because of (32b), (35) can be written, equivalently, as

$$\frac{B}{\pi i} \left[\frac{(\kappa_2 \mu_1 + \mu_2)F'^+(x)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)} + \frac{(\kappa_1 \mu_2 + \mu_1)F'^-(x)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)} \right] = p(x), \quad a < x < b, \quad (36)$$

and, further, because of (32), as

$$\frac{B}{2\pi i} \left[\left(1 + \frac{A}{B}\right)F'^+(x) + \left(1 - \frac{A}{B}\right)F'^-(x) \right] = p(x), \quad a < x < b. \quad (37)$$

Next, by taking into account the fundamental Plemelj formula (18), as well as the definition (34) of the constant c , we observe directly that (37) reduces to (33) (in the special case where $k(t, x) \equiv 0$ under consideration).

Finally, we can add that in the present application it is clear from (32) and (34) that we have used a complex (rather purely imaginary) value for c . It can easily be seen that this has no influence on the results of Sections 4, 5 and 6, but, evidently, the fundamental definition (2) of generalized Mangler-type principal value integrals is no more applicable. Equation (4) can be used as the definition of this class of integrals in this case.

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