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# Detection of loading/geometrical singularities in isotropic elastic media with the use of Gröbner bases

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**Abstract** The method of Gröbner bases in computer algebra is applied to the detection and location of loading or geometrical singularities inside two- or three-dimensional isotropic elastic media. The approach consists in using available experimental data concerning the stress/strain components away from the singularity in order to decide whether these components verify the relations corresponding to the particular singularity or not. These conditions can frequently be obtained by the method of Gröbner bases and the related Buchberger algorithm. In the case of an affirmative conclusion, we can further proceed to the location of the singularity. Alternatively, the present approach yields compatibility equations for the stress/strain components in elastic media with loading/geometrical singularities. These equations are displayed in detail for some common stress fields corresponding to concentrated forces as well as to a crack tip.

**Keywords** Elasticity · Stresses and strains · Compatibility equations · Concentrated forces · Cracks · Detection/location of singularities · Loading/geometrical singularities · Elimination algorithms · Gröbner bases · Buchberger's algorithm · Computer algebra



## 1. Introduction

The stress/strain fields in classical two-/three-dimensional isotropic elasticity satisfy well-known equations, which assure the compatibility of their components and they can be called compatibility equations. In two-dimensional (plane) elasticity, such equations are the two equilibrium equations for the stress components and the classical compatibility equation for the strain components yielding a similar equation for the stress components [1, pp. 27–31]. Similarly, in three-dimensional

[1]Both the internal and the external links (all appearing in blue) were added by the author on December 29, 2017 for the online publication of this technical report.

elasticity, we have also the three partial differential equations of equilibrium for the stress components and the six analogous compatibility equations for the strain components easily yielding similar equations for the stress components on the basis of the fundamental Hooke's law [1, pp. 235–239]. Alternatively, the three equilibrium equations for the stress components can be expressed in terms of the displacements [1, pp. 240–242] and so on. All of the aforementioned equations are partial differential equations.

It is an interesting task to experimentally verify the above compatibility equations by employing the classical techniques of experimental stress analysis such as strain gauges, photoelasticity, interferometry, Moiré fringes, holography, speckle patterns, their various extensions and combinations, etc. These techniques are described in detail in the related books (see, e.g., [2]) and in several journals including *Experimental Mechanics* and *Experimental Techniques*. Since the stress and the strain components are related in elasticity through Hooke's law, in the sequel we will assume that we are working with the stress components only although our results can easily be written in terms of the strain components too.

Here we propose one more kind of compatibility equations for two- and three-dimensional problems in classical isotropic elasticity: we suggest to consider an isotropic elastic medium with a singularity either in loading (such as a concentrated force, a couple of forces, a discontinuity of the loading along the boundary of the medium, etc.) or in geometry (such as a crack tip, a V-notch, a circular or elliptical hole, etc.). In all of these cases, having available the components of the related stress fields from the books on elasticity (see e.g. [1–3]) and/or the thousands of available related papers, we can eliminate the Cartesian coordinates $x$ and $y$ in two dimensions and $x$, $y$ and $z$ in three dimensions and obtain more or less simple compatibility equations for the stress components, but not of general validity as previously: now their validity will hold true only for the loading/geometrical discontinuity having been assumed in advance.

In this way, we can use again the techniques of experimental mechanics in order to verify the aforementioned compatibility equations, which are completely irrelevant to the equilibrium equations and the compatibility conditions already mentioned at the beginning of this section and, moreover, they are ordinary (purely algebraic) equations and not partial differential equations. But, of course, much more important is to use this approach in order to detect the presence or the absence of the singularity by using available experimental information at points probably away from the singularity, inserting these into the compatibility equations just mentioned and verifying or rejecting the conclusion about the presence or the absence of the singularity on the basis of whether or not the compatibility equations are verified. It seems that this approach has not been used so far to this author's best knowledge. Of course, after the detection of a singularity, we can proceed to its location with respect to a system of Cartesian coordinates fixed on the points where the experimental data were collected.

Here we will restrict ourselves to the theoretical part of the present approach, that is to the construction of the compatibility equations for some common stress fields by using the method of Gröbner bases in computer algebra and we will leave the experimental part to the interested reader or researcher.

The computational algorithm (Buchberger's algorithm) for the construction of Gröbner bases in the theory of multivariate algebraic polynomials was proposed for the first time by B. Buchberger in his Ph.D. thesis at the University of Innsbruck (appeared in 1965). Buchberger's original and fundamental algorithm was further studied and extended by him as well as by many other interested researchers in the field. Today all major computer algebra systems incorporate a package (or, simply, single commands) for Gröbner bases, whose name was attributed to these bases by Buchberger in 1976 in honour of his advisor, W. Gröbner, Professor at the University of Innsbruck and with fundamental contributions to the related areas (see, e.g., [4]). As far as Gröbner bases are concerned, we can make reference to the two most basic review papers by Buchberger [5, 6], where several related references can be found. Moreover, the books [7–12] include sections or whole chapters devoted to Gröbner bases. Finally, the books [13] and [14] are essentially completely devoted to Gröbner bases, elimination (as is here the case), applications and related fields.

Here we will use the computer algebra system *Maple V* [15–17] and, in particular, its package for Gröbner bases [17, pp. 469–478]. This package is an efficient one and friendly in its use, making use of the Buchberger algorithm, although it cannot be claimed that it is the best one available. Similar packages are included in *Reduce* [8], *Axiom* [18] and *Mathematica* [19]. The latest version of *Mathematica* (version 3, [19]) contains a very efficient implementation of the Buchberger algorithm, probably the *Mathematica* implementation for Gröbner bases having been prepared by Buchberger [20]. The Buchberger algorithm has been also implemented in the C-language-based *SACLIB* computer algebra system [21] by Windsteiger [22, 23].

Previous related results in applied mechanics by the author and Anastasselou (by using Gröbner bases) can be found in Refs. [24–34]. Strangely enough, no further references related to the applications of Gröbner bases to applied mechanics problems are known to the present author, of course beyond the application of these bases to inverse robot kinematics [6, 13] and to mechanisms [35]. In fact, the present application (detection and location of a singularity in an isotropic elastic medium) is also an inverse problem exactly as has been the case in the problem of "design" of trusses having been studied in brief in Ref. [28].

## 2. Detection of singularities—the approach

For the detection of singularities we have to use experimental data for the stress (or, alternatively, the strain) components available at one point of the isotropic elastic medium and check whether the compatibility equation(s) for these components is (are) completely verified or not. In the first case, most probably, this singularity in the elastic medium (either in loading or in geometry) is present. If we wish, we can use experimental data for one or few additional points of the isotropic elastic medium in order to be completely sure about our conclusion. In the next section, we will display concrete compatibility equations having been derived by using the Buchberger's algorithm implementation in *Maple V* (for the computation of Gröbner bases) for five typical problems in elasticity incorporating singularities and including the crack tip problem in fracture mechanics. The whole approach is very simple and will be described just below.

At first, for two-dimensional problems, we initially assume that we know the theoretical expressions for the stress components $\sigma_x$, $\sigma_y$ and $\sigma_{xy}$ at the points about a singularity (either a loading singularity such as a concentrated force, a pair of concentrated forces, a moment, etc. or a geometrical singularity such as a dislocation, a circular hole, an elliptical hole, similar inclusions, a crack tip, a V-notch tip, corner points, etc.). Evidently, these expressions will include the Cartesian coordinates $(x, y)$ or, better, just for convenience, the polar coordinates $(r, \theta)$ and, probably, material parameters such as the Poisson ratio $\nu$ and, moreover, loading parameters such as the applied load $F$, the applied moment (couple of forces) $M_0$, the stress intensity factor $K$, etc. Of course, it is also convenient to assume that the origin $O$ of the Cartesian or the polar coordinate system coincides with the singularity itself (or with its centre in the case of holes). Then we will have available the related formulae [1, 3]

$$\begin{aligned} \sigma_x &= \sigma_{x,1}(x, y, C) = \sigma_{x,2}(r, \theta, C), \\ \sigma_y &= \sigma_{y,1}(x, y, C) = \sigma_{y,2}(r, \theta, C), \\ \sigma_{xy} &= \sigma_{xy,1}(x, y, C) = \sigma_{xy,2}(r, \theta, C), \end{aligned} \tag{1}$$

where $C$ denotes the set of parameters involved in these formulae, e.g. $C \subseteq \{\nu, F, M_0, K, \dots\}$.

If the available expressions (1), including the Cartesian coordinates $(x, y)$, are polynomials in $x$, $y$ and $C$ (as is sometimes the case), then we can consider the related set $P$ of polynomials (better polynomial equations)

$$\begin{aligned} P_1(x, y, \sigma_x, C) &= \sigma_x - \sigma_{x,1}(x, y, C) = 0, \\ P_2(x, y, \sigma_y, C) &= \sigma_y - \sigma_{y,1}(x, y, C) = 0, \\ P_3(x, y, \sigma_{xy}, C) &= \sigma_{xy} - \sigma_{xy,1}(x, y, C) = 0 \end{aligned} \tag{2}$$

(with $P = \{P_1, P_2, P_3\}$) in the list of variables (including the set of parameters $C$)

$$V = [x, y, \sigma_x, \sigma_y, \sigma_{xy}, C]. \tag{3}$$

Now it is generally possible to eliminate the Cartesian coordinates $x$ and $y$ (equivalently, the polar coordinates $r$ and $\theta$) in the set of polynomials $P$ by using any elimination method. Then we will find a fourth polynomial equation

$$Q(\sigma_x, \sigma_y, \sigma_{xy}, C) = 0, \tag{4}$$

which will be our compatibility equation for the stress components, generally incorporating also the parameters $C$ involved or, at least, the material ones, mainly the dimensionless Poisson ratio $\nu$.

In extremely simple cases, the above elimination can be performed by hand. Let us consider, e.g., the case of a moment (concentrated couple of forces) $M_0$ acting at the origin $O$ of the Cartesian coordinate system $Oxy$ in plane elasticity for an infinite medium. Then the stress components $\sigma_x$, $\sigma_y$ and $\sigma_{xy}$ can be easily determined in polar coordinates $(r, \theta)$ from the related available equations [1] as

$$\sigma_x = -g \sin 2\theta, \quad \sigma_y = g \sin 2\theta, \quad \sigma_{xy} = g \cos 2\theta, \tag{5}$$

where $g$ denotes the quantity

$$g = -\frac{M_0}{2\pi r^2}. \tag{6}$$

By adding the first two of Eqs. (5), we can easily eliminate both polar coordinates $r$ and $\theta$ and find the sought (here extremely elementary) compatibility condition, which is simply

$$\sigma_x + \sigma_y = 0. \tag{7}$$

Moreover, through appropriate divisions, again in Eqs. (5), we find that the orientation $\theta$ of our observation point $M$ with respect to the center $O$ of the Cartesian coordinates (or conversely) can be determined from

$$\tan 2\theta = -\frac{\sigma_x}{\sigma_{xy}} \quad \text{or} \quad \tan 2\theta = \frac{\sigma_y}{\sigma_{xy}}. \tag{8}$$

Finally, by adding the squares of the first and the third (or the second and the third) of the same equations, we observe that the mixed loading–geometrical parameter $g$ in our problem can be determined from

$$g^2 = \sigma_x^2 + \sigma_{xy}^2 \quad \text{or} \quad g^2 = \sigma_y^2 + \sigma_{xy}^2. \tag{9}$$

Here the elementary trigonometric identity

$$\cos^2 2\theta + \sin^2 2\theta = 1 \tag{10}$$

has been taken into account, supplementing Eqs. (5).

Although we have been successful in performing the required eliminations in the elementary plane elasticity problem of a moment $M_0$ through manual computations, unfortunately, in general, by no means is this the case and the use of specialized computer algebra elimination algorithms becomes indispensable. Such algorithms are described in sufficient detail in the aforementioned references (a related interesting introduction to these algorithms is also made in Ref. [36]) and include: (i) classical resultants (see, e.g., Refs [7, 10–12]), (ii) the Ritt–Wu characteristic set method (see, e.g., Refs [11, 37, 38]) or the alternative Wang elimination method [39] and, of course, (iii) Gröbner bases (the Buchberger classical algorithm) with several related references already mentioned in the previous section. The latter algorithm is the elimination algorithm that will actually be used below because it seems to be more "systematic" than the first two elimination algorithms in the sense that the derived set of polynomials $G$ (from the set of original polynomials $P$) is *completely equivalent* to $P$ as far as their (real and complex) solutions (the solution sets) are concerned. Frequently, this is not the case with the first two elimination algorithms (resultants and characteristic sets). We can also mention that the characteristic set elimination algorithm has been implemented in *Maple* by Wang [40]. This implementation was recently used by Anastasselou [41] for the derivation of compatibility equations in differential/difference form for beam problems in structural mechanics.

We can also add that when using the Buchberger algorithm concerning Gröbner bases as an elimination tool, we should do so with the lexicographic term ordering option in the sense that we should ask that the variables in the list $V$ in Eq. (3) be eliminated in the order that they appear in $V$. This will ensure the elimination of the Cartesian coordinates $x$ and $y$ so that at first we can reach our compatibility equation $Q = 0$ in Eq. (4). (This is not the case e.g. in the total degree term ordering in Gröbner bases.)

It is also well known that the final set $G$ of polynomials (equivalently, essentially polynomial equations) in the resulting Gröbner basis (whose last element was assumed to be the compatibility condition $Q$) is more or less "triangular" in the sense that the last element of $G$ is free from both $x$ and $y$ (in our application), the previous element (or elements) is (are) free from $x$ (it include(s) only $y$) and the first element (or elements) include(s) both $x$ and $y$, a situation somewhat analogous to that appearing during the solution of linear algebraic equations with the help of the Gauss classical elimination algorithm. This means that beyond the compatibility condition $Q = 0$, we will *simultaneously* have available appropriate polynomial equations for the determination of $y$ and, next, of $x$, that is for the complete location of the singularity, evidently with respect to the observation point $M$, where our experimental data for the stress components have been gathered. At this point, it can also be emphasized that no element of the Gröbner basis $G$ is useless; none of them can be ignored both when deciding on the existence of the singularity (the elements concerning the compatibility equations although in two dimensions, generally, just one condition appears) and when locating such a singularity (the elements which can be used for the determination of the related position, Cartesian/polar variables) in the case of an affirmative decision.

Of course, it is understood that frequently the above approach should be somewhat modified. This is the case when the Cartesian coordinates $x$ and $y$ do not appear in such an ideal, polynomial way. Then we must use other appropriate variables instead of $x$ and $y$ (e.g. the variables $u = y/x$ and $v = (x^2+y^2)^{-1/2}$) so that our set $P$ in Eqs. (2) can be really a set of polynomials $P_i$ ($i = 1, 2, 3$).

Quite similar is the case with the polar coordinates $r$ and $\theta$ as will be seen in the concrete applications of the next section. But in the case of polar coordinates, one more problem arises: in fact, the polar angle $\theta$ generally appears through the sine and cosine trigonometric functions. Therefore, we have essentially two "independent" variables, i.e.

$$c_\theta = \cos\theta \quad \text{and} \quad s_\theta = \sin\theta \quad \text{whence} \quad P_4 = c_\theta^2 + s_\theta^2 - 1 = 0 \tag{11}$$

(although in the crack problem of Subsection 3.4 the situation is slightly more complicated). The latter classical formula is our fourth polynomial, $P_4$, in the present case, where our list of variables $V$ takes the following modified form:

$$V_p = [s, c_\theta, s_\theta, \sigma_x, \sigma_y, \sigma_{xy}, C] \tag{12}$$

with $s$ referring to an appropriate application-dependent power of the polar radius $r$, e.g. (in the applications of the next section)

$$s = 1/r, \quad s = 1/r^2 \quad \text{or} \quad s = 1/\sqrt{r}. \tag{13}$$

Alternatively, instead of using the trigonometric identity $P_4$ in Eqs. (11), we can simply express both $\cos\theta$ and $\sin\theta$ in terms of the tangent of $\theta/2$, $\tan(\theta/2)$, employed now as a new variable $t$ [13]. But it is doubtful whether this alternative, which reduces both the number of variables in $V_p$ and the number of the polynomials in $P$ by one (now $P_4$ is useless) is preferable or not.

Under these precautions, the application of the Buchberger algorithm to the final set of polynomials

$$P_0 = P \cup P_4 = \{P_1, P_2, P_3, P_4\} \tag{14}$$

(in the case of polar coordinates) will yield the related Gröbner basis $G$, the last polynomial element of which will be the required compatibility equation. On the basis of this condition and the experimentally gathered data about the stress components at the point $M$, we will be able to decide about the existence or absence of the singularity in a concrete application. Next, in the first case, we will be able to use the previous elements of the Gröbner basis $G$ (from the end towards the beginning in this basis) in order to determine the location of the singularity, e.g., in polar coordinates, by calculating $c_\theta$ and $s_\theta$ in the case of the list of variables $V_p$ in Eq. (12). Then the polar coordinates $r$ and $\theta$ will be directly available as well and the present inverse elasticity problem will have been completely solved. It will also be possible to verify the resulting solution in the case that we have available experimental data at other points of the isotropic elastic medium as well.

The above approach can also directly be generalized to three-dimensional isotropic elasticity problems. In this case, we will have six stress components

$$\Sigma = \{\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\} \tag{15}$$

and three position variables to eliminate such as the Cartesian coordinates $x$, $y$ and $z$ or the spherical coordinates $\rho$, $\theta$ and $\phi$. Therefore, in principle, we will determine a Gröbner basis $G$ with three or more compatibility equations (as will be the case in the application of Subsection 3.5 below) unless we choose to use only four of the above stress components $\Sigma$. Of course, we will have again to use auxiliary variables and/or trigonometric identities in order to be able to work with the set of polynomials $P$, which will yield the Gröbner basis $G$, permitting us to decide about the existence of the singularity and, in such a case, to locate its position working in a way completely analogous to that in the two-dimensional case.

We can mention that in case that we have more than one compatibility equation available (this is generally the case in three dimensions), we can directly create additional (simpler or more complex) compatibility equations. This can be achieved simply by constructing any polynomial linear combination (not including, of course, position variables) of the available polynomial compatibility equations, i.e. the subset $G_\sigma$ including the last elements of the Gröbner basis $G$ under the lexicographic term ordering used, which will be free from the Cartesian or polar or other auxiliary position coordinates.

Finally, it is clear that we can move from stress to strain components and conversely at will because of Hooke's law in linear elasticity [1, 3]. The choice of stress or strain components may also depend on the type of the available experimental data.

Now we will illustrate the present approach by means of five concrete applications concerning loading or geometrical singularities, four in two dimensions and one in three dimensions.

## 3. Elasticity applications

### 3.1. *A normal concentrated force at a point of a straight boundary*

First, we consider the problem of a normal concentrated force $F$ at a point $O$ of a straight boundary of an isotropic elastic half-plane. The stress components are given by the formulae [1, pp. 97–99]

$$\sigma_x = -g\cos^4\theta, \quad \sigma_y = -g\sin^2\theta\cos^2\theta, \quad \sigma_{xy} = -g\sin\theta\cos^3\theta, \tag{16}$$

where $g$ is given by

$$g = \frac{2F}{\pi a}. \tag{17}$$

In Eqs. (16) $\theta$ denotes the angle between the direction of the concentrated force $F$ (normal to the straight boundary and applied at a point $O$ of this boundary) and the direction $OM$, where $M$ is the point $(x, y)$ of measurement of the stress components. Moreover, in Eq. (17) $a$ denotes the distance of $M$ from the straight boundary.

In this way, we have available three equations, Eqs. (16), and, therefore, we can eliminate both $g$ and $\theta$ in order to derive the compatibility equation for the detection of the presence or absence of the concentrated force $F$ (whose magnitude is assumed not to be known in advance). For the solution of this elimination problem we used *Maple*'s Gröbner bases package [17, pp. 469–478] and, more explicitly, the related command concerning the determination of Gröbner bases under the pure lexicographic term ordering option so that the undesired variables can be eliminated. Further related computational details can be found in [28]. Our "hypotheses" have been the three aforementioned Eqs. (16) plus the elementary trigonometric equation

$$\cos^2\theta + \sin^2\theta = 1, \tag{18}$$

which (as was already mentioned in the previous section) is necessary since Eqs. (16) are not simply multivariate polynomials, but they also include trigonometric functions.

By using Eqs. (16) and (18) with the list of variables

$$V = [\cos\theta, \sin\theta, g, \sigma_x, \sigma_y, \sigma_{xy}] \tag{19}$$

(in this order) we eliminated $\cos\theta$, $\sin\theta$ and $g$ and, therefore, we obtained the required compatibility equation for the stress components in our problem, which is simply

$$\sigma_x\sigma_y = \sigma_{xy}^2. \tag{20}$$

For additional clarity, below we display the *Maple V* commands used to derive Eq. (20)

```
> tx:=-g*cs^4: ty:=-g*sn^2*cs^2: txy:=-g*sn*cs^3:
> h1:=sx-tx: h2:=sy-ty: h3:=sxy-txy: h4:=cs^2+sn^2-1:
> grobner[gbasis]([h1,h2,h3,h4],[cs,sn,g,sx,sy,sxy],plex);
```

The resulting Gröbner basis consisted of ten elements (multivariate polynomials), of which we were interested only in the last element, which is

```
-sxy^2+sx*sy
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In Eq. (20) we rewrote this element as our fundamental compatibility equation that must be satisfied in the present application by the stress components at any point of the elastic half-plane including, of course, the point where we gathered our experimental data for the stress components.

If Eq. (20) is satisfied at one or, what is even better, several points of the elastic half-plane, then we can use additional elements of the above Gröbner basis or, better, the resulting equivalent equations. For example, in our application we can use the following equation (directly resulting from the ninth element of the above Gröbner basis)

$$g = -\frac{\sigma_x \sigma_{xy}^2 + 2\sigma_y \sigma_{xy}^2 + \sigma_y^3}{\sigma_{xy}^2} \tag{21}$$

in order to determine $g$ and, if we know the distance $a$ of the point $M$, where we have measured the stress components, from the boundary of the half-plane, we have available the magnitude of the concentrated force $F$ (taking always into account Eq. (17)). Alternatively, $g$ can also be obtained from the eighth element,

$$g = -\frac{\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}^2}{\sigma_x}, \tag{22}$$

of the aforementioned Gröbner basis.

Probably, more important is the location of the point $O$ on the boundary of the half-plane at which the concentrated force $F$ is applied. In order to completely determine this point, we need to determine $\sin\theta$ and $\cos\theta$. There are several equations for these quantities directly resulting from the available "polynomials" in our Gröbner basis, e.g.

$$\sin^2\theta = \frac{\sigma_y}{\sigma_x + \sigma_y} \quad \text{or} \quad \sin^2\theta = \frac{\sigma_y^2}{\sigma_y^2 + \sigma_{xy}^2}. \tag{23}$$

Analogous equations can also be derived for $\cos\theta$ by using an appropriate ordering in the variables of the Gröbner basis or simply from Eq. (18). Of course, in the case of doubt about a sign, we can use one more point $N$ (beyond the original point $M$) of the elastic half-plane for the determination of the angle $\theta$ but this is just a detail.

Our conclusion is that the present approach permitted us to decide whether or not our elastic stress field in a half-plane is due to a normal concentrated force acting on the boundary of the half-plane with simple measurements of the stress components at one or two interior points of the half-plane. In an affirmative case, we can next determine both the magnitude $F$ of the concentrated force (assuming that we know the distance $a$ of the point of measurement $M$ from the boundary) as well as the exact position of the point $O$ of application of $F$ with respect to $M$ (by using the angle $\theta$). Therefore, the whole problem of detection and location of the concentrated force $F$, which is a simple loading singularity in the present application, has been completely solved.

### 3.2. *A concentrated force in an infinite plane*

This is a slightly more difficult problem from the computational point of view. We consider an infinite plane isotropic elastic medium, where a concentrated force $F$ is applied at a point $O$.

At a point $M$ of the elastic medium the stress components are given by the following formulae [1, pp. 127–129]:

$$\begin{aligned}\sigma_x &= g\cos\theta\,[-(3+\nu)+2(1+\nu)\sin^2\theta],\\ \sigma_y &= g\cos\theta\,[1-\nu-2(1+\nu)\sin^2\theta],\\ \sigma_{xy} &= -g\sin\theta\,[1-\nu+2(1+\nu)\cos^2\theta],\end{aligned} \tag{24}$$

where $\nu$ denotes the Poisson ratio of the isotropic material of the elastic medium and the variable $g$ is now given by

$$g = \frac{F}{4\pi r}, \tag{25}$$

where $r$ is the distance of the point $M$ from the point $O$ and $\theta$ is the angle of $OM$ to the direction of $F$.

By using Eqs. (24), together with Eq. (18) again, as the "hypotheses" (polynomials) for our Gröbner basis in a way analogous to that in the previous application and employing the following list of variables:

$$V = [g, \cos\theta, \sin\theta, \sigma_x, \sigma_y, \sigma_{xy}], \tag{26}$$

we found easily our Gröbner basis, whose last element (in the case of a purely lexicographic ordering of the variables as is assumed again here and in the sequel) can be considered (and rewritten in a slightly modified form) as our compatibility equation for the stress components. This equation has the form

$$(-1+\nu)(\sigma_x^3+\sigma_y\sigma_{xy}^2)+(3\nu+1)\sigma_x\sigma_{xy}^2-(3+\nu)\sigma_x^2\sigma_y = 0. \tag{27}$$

(In fact, Gröbner bases concern only polynomials and, therefore, the equals sign and the right-hand side, zero, should always be added by us. This has been the case in Eq. (27) as well.)

The variables in Eq. (26) have been ordered differently from Eq. (19). This is not particularly important since the resulting compatibility condition, Eq. (27), is exactly the same as it should be. But on the other hand, the Gröbner basis based on Eq. (26) consists just of 9 elements, whereas that based on Eq. (19) consists of 17 elements and, therefore, in principle, it is not preferable. Similarly, in the first case the computer used required 13 s whereas in the second case it required 15 s. Of course, on the other hand, the second Gröbner basis, i.e. that based on the ordering (19), permits the direct determination of the quantity $g$ (from its 16th element). The related formula has the form

$$g^2 = \frac{(\nu-1)(\sigma_x^2+\sigma_y^2)-4\sigma_x\sigma_y+2(\nu+1)\sigma_{xy}^2}{2(\nu+1)(\nu-1)^2}, \tag{28}$$

which, evidently, cannot be derived from the first Gröbner basis, i.e. that based on the list (26).

In the case when the stress components satisfy the compatibility condition (27), we can proceed to locate the point $O$ of application of the concentrated force $F$. To this end, we can use the penultimate element of the first of the aforementioned Gröbner bases, which can be directly written in the form of the following equation:

$$\sin^2\theta = -\frac{(1-\nu)\sigma_x^2-(3+\nu)\sigma_y^2+2(1+\nu)(\sigma_x\sigma_y-\sigma_{xy}^2)}{2(1+\nu)(\sigma_y^2+\sigma_{xy}^2)}. \tag{29}$$

(One more analogous equation, again for $\sin^2\theta$, is also available in the same Gröbner basis.)

Of course, we need more than one point of "observation" of the stress components $M$ in order to completely determine $\theta$ and further determine $O$ by using two values, $\theta_1$ and $\theta_2$, of the angle $\theta$

for two points $M_1$ and $M_2$. After having determined $O$ by using the distance and the orientation of $M_1M_2$ and the aforementioned two angles $\theta_{1,2}$, we are able to determine $g$ by using one more appropriate element of our Gröbner basis or, alternatively, Eq. (28), based on the variables ordering (19) with the list of variables $V$ appropriately reordered so that $g$ can be directly determined in terms of the stress components. This is not the case with the ordering of the variables in Eq. (26). We will not display the related formula for the variables ordering (26) but we can mention that the determination of $g$ after the determination of $O$ with respect to $M_{1,2}$ permits us to directly evaluate the magnitude of $F$ from Eq. (25). Alternatively, $g$ can be determined directly from Eqs. (24). We will not present these details here.

### 3.3. *A pair of concentrated forces in an infinite plane*

This is an even more difficult application from the computational point of view and we do not think that the related extremely complicated computations could ever be undertaken by humans. We consider two equal and opposite forces $F$ acting at the points $O$ and $O'$ of an infinite plane isotropic elastic medium with the very small distance $OO'$ equal to $d$. Then the formulae for the stress components are [1, pp. 129–130]:

$$\begin{aligned} \sigma_x &= g[-(3+\nu)\cos^2\theta + (1-\nu)\sin^2\theta + 8(1+\nu)\sin^2\theta\cos^2\theta], \\ \sigma_y &= g[(1-\nu)\cos^2\theta + (1+3\nu)\sin^2\theta - 8(1+\nu)\sin^2\theta\cos^2\theta], \\ \sigma_{xy} &= g[-(6+2\nu) + 8(1+\nu)\sin^2\theta]\sin\theta\cos\theta, \end{aligned} \tag{30}$$

where the symbols (polar coordinates) $r$ and $\theta$ have meanings analogous to those in the preceding application and the variable $g$ is defined in the present application by

$$g = \frac{Fd}{4\pi r^2}\,. \tag{31}$$

The Gröbner basis in the present application is sufficiently complicated and we will display only its last element (obtained in a way analogous to that used in the previous two applications and again with the purely lexicographic term ordering option in *Maple*'s related command), that is the compatibility equation for the stress components under the present "singularity" in the stress field. This equation, having, of course, again the Poisson ratio $\nu$ as a parameter, is (as this was derived by the computer with the equals sign and the right-hand side, zero, added once more by us)

$$\begin{aligned} &(-16\nu^4 - 64\nu^3 - 96\nu^2 - 64\nu - 16)\sigma_{xy}^4 \\ &\quad + (24\nu^4 + 16\nu^3 + 32\nu^2 + 48\nu + 72)\sigma_y^2\sigma_{xy}^2 \\ &\quad + (-\nu^4 - 8\nu^3 - 18\nu^2 + 27)\sigma_y^4 \\ &\quad + (96\nu^4 + 128\nu^3 + 192\nu^2 + 128\nu + 96)\sigma_x\sigma_y\sigma_{xy}^2 \\ &\quad + (72\nu^4 + 48\nu^3 + 32\nu^2 + 16\nu + 24)\sigma_x^2\sigma_{xy}^2 \\ &\quad + (27\nu^4 - 18\nu^2 - 8\nu - 1)\sigma_x^4 \\ &\quad + (-18\nu^4 - 120\nu^3 - 236\nu^2 - 120\nu - 18)\sigma_x^2\sigma_y^2 \\ &\quad + (-8\nu^4 - 56\nu^3 - 120\nu^2 - 72\nu)\sigma_x\sigma_y^3 \\ &\quad + (-72\nu^3 - 120\nu^2 - 56\nu - 8)\sigma_x^3\sigma_y = 0. \end{aligned} \tag{32}$$

Of course, although we will not display here any related equations, we can also obtain $\sin\theta$, $\cos\theta$ and $g$ by using additional elements of the Gröbner basis beyond its last element, the compatibility Eq. (32), which will permit us (by using two or three points as "measurement" points for the stress components) not only to detect the singularity (now a pair of concentrated forces), but also to completely locate it (the points $O$ and $O'$ of application of the opposite forces) and, finally, determine its "strength" $Fd$ by taking into consideration Eq. (31).

### 3.4. *A single crack tip*

Crack problems and fracture mechanics constitute a very important topic in the classical theory of elasticity. This application concerns the simple case of a single straight crack in an infinite plane isotropic elastic medium under mode I conditions with a stress intensity factor equal to $K$ at the crack tip $O$. We assume no influence of the second crack tip (if its exists) and we will use the first terms of the classical (in fracture literature) Williams' expansions for the stress components near the crack tip [42, p. 24], [43, p. 11]

$$\begin{aligned} \sigma_x &= g\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \\ \sigma_y &= g\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \\ \sigma_{xy} &= g\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \end{aligned} \tag{33}$$

with $(r,\theta)$ again denoting the polar coordinates referring to the crack tip $O$ under consideration with the crack itself lying along the negative $Ox$-axis and, further, the variable $g$ being now given by

$$g = \frac{K}{\sqrt{2\pi r}}. \tag{34}$$

First, we observe that in the present application we wish to detect and locate a geometric discontinuity, the crack tip $O$, and not a loading discontinuity as in the previous three applications. Secondly, beyond the fundamental angle $\theta/2$ (in sine and cosine terms), where Eq. (18) (but here with $\theta/2$ instead of $\theta$) should be taken into account, we have also the dependent values of the sine and cosine of $3\theta/2$. Therefore, we must also use (beyond Eqs. (33) and (18) as was explained previously) the additional well-known trigonometric identities

$$\sin\frac{3\theta}{2} = 3\sin\frac{\theta}{2} - 4\sin^3\frac{\theta}{2}, \quad \cos\frac{3\theta}{2} = 4\cos^3\frac{\theta}{2} - 3\cos\frac{\theta}{2} \tag{35}$$

inside the command for the derivation of the Gröbner basis, which, in this way, will incorporate six equations (three for the stress components and three auxiliary equations for the necessary classical trigonometric identities as was already mentioned).

The last element of the derived related Gröbner basis in the present application (by commands analogous to those having been already used in the previous applications and again with *Maple V*) is the following one:

$$-10(\sigma_x^2+\sigma_y^2)\sigma_{xy}^2 - 15\sigma_x^2\sigma_y^2 + 7\sigma_x\sigma_y^3 - \sigma_y^4 + 44\sigma_x\sigma_y\sigma_{xy}^2 + 9\sigma_x^3\sigma_y - 32\sigma_{xy}^4 = 0. \tag{36}$$

This is our compatibility equation for the stress components in the present application. Additional equations of the derived Gröbner basis permit us the complete location of the crack tip $O$ as well as the determination of the stress intensity factor $K$ by using measurements of the stress components only at two or three points $M$ of the elastic field.

To become somewhat more explicit, just in this particular application, we will display below the whole Gröbner basis for the crack problem under consideration. This basis, $G$, consists of $n = 16$ polynomials in the list of variables

$$V = [g, c_\theta, s_\theta, \sigma_x, \sigma_y, \sigma_{xy}], \tag{37}$$

where $c_\theta$ and $s_\theta$ are now defined by

$$c_\theta = \cos(\theta/2), \quad s_\theta = \sin(\theta/2), \tag{38}$$

it is based on the three identities (33), directly written in equivalent polynomial forms with the help of Eqs. (35), supplemented by the obvious trigonometric polynomial $P_4 = c_\theta^2 + s_\theta^2 - 1 = 0$ (i.e. four polynomials $P_i$ are used) and has the form

$$G_1 = 2c_\theta g - \sigma_x - \sigma_y = 0, \tag{39}$$
$$G_2 = 2(s_\theta^2 - 1)g + (\sigma_x + \sigma_y)c_\theta = 0, \tag{40}$$
$$G_3 = 6\sigma_y g + (\sigma_x + 5\sigma_y)(-\sigma_y c_\theta + \sigma_{xy} s_\theta) + 4(\sigma_x + \sigma_y)\sigma_{xy} s_\theta^3 = 0, \tag{41}$$
$$G_4 = 4\sigma_{xy} g + (3\sigma_y^2 - 10\sigma_x\sigma_y + 3\sigma_x^2 + 16\sigma_{xy}^2)s_\theta = 0, \tag{42}$$
$$G_5 = c_\theta^2 + s_\theta^2 - 1 = 0, \tag{43}$$
$$G_6 = [(8s_\theta^2 - 3)\sigma_y + 3\sigma_x]c_\theta + 2(5 - 4s_\theta^2)s_\theta\sigma_{xy} = 0, \tag{44}$$
$$G_7 = (3\sigma_x - \sigma_y)c_\theta s_\theta + 2(2s_\theta^2 - 1)\sigma_{xy} = 0, \tag{45}$$
$$G_8 = (3\sigma_x^2 + \sigma_y^2 - 4\sigma_x\sigma_y)c_\theta - 8(\sigma_x + \sigma_y)\sigma_{xy}s_\theta^3 + 2(5\sigma_x + \sigma_y)\sigma_{xy}s_\theta = 0, \tag{46}$$
$$G_9 = \sigma_{xy}c_\theta + (\sigma_x + \sigma_y)s_\theta^3 - \sigma_x s_\theta = 0, \tag{47}$$
$$G_{10} = (\sigma_x + \sigma_y)(4s_\theta^2 - 3)s_\theta^2 - \sigma_x + \sigma_y = 0, \tag{48}$$
$$G_{11} = 8(\sigma_y^2 + \sigma_{xy}^2)(8s_\theta^2 - 5)s_\theta^2 - 9(\sigma_x - \sigma_y)\sigma_y - 14\sigma_{xy}^2 = 0, \tag{49}$$
$$G_{12} = 4(\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}^2)s_\theta^2 - 4\sigma_x^2 - 3\sigma_y^2 + 7\sigma_x\sigma_y - 14\sigma_{xy}^2 = 0, \tag{50}$$
$$G_{13} = 8(\sigma_x\sigma_y - \sigma_{xy}^2)s_\theta^2 + \sigma_x\sigma_y - \sigma_y^2 - 2\sigma_{xy}^2 = 0, \tag{51}$$
$$G_{14} = 8[(\sigma_x + 2\sigma_y)\sigma_{xy}^2 + \sigma_y^3]s_\theta^2 - 9\sigma_x^2\sigma_y + (15\sigma_y^2 + 2\sigma_{xy}^2)\sigma_x - 6\sigma_y^3 - 28\sigma_y\sigma_{xy}^2 = 0, \tag{52}$$
$$G_{15} = 8(\sigma_y^2 + \sigma_{xy}^2)^2 s_\theta^2 - 9\sigma_x^2\sigma_y^2 + (15\sigma_y^2 + \sigma_{xy}^2)\sigma_x\sigma_y - 27\sigma_y^2\sigma_{xy}^2 - 6\sigma_y^4 + 2\sigma_{xy}^4 = 0, \tag{53}$$
$$G_{16} = 9\sigma_x^3\sigma_y - 5(3\sigma_y^2 + 2\sigma_{xy}^2)\sigma_x^2 + (7\sigma_y^2 + 44\sigma_{xy}^2)\sigma_x\sigma_y - \sigma_y^4 - 10\sigma_y^2\sigma_{xy}^2 - 32\sigma_{xy}^4 = 0. \tag{54}$$

Of course, the last equation $G_{16}$, coinciding with its equivalent form (36), is our compatibility condition. It is understood that the Gröbner basis algorithm (the Buchberger algorithm) in computer algebra systems (such as in *Maple V* here) does not explicitly introduce the "$= 0$" sign, leaving the above elements $G_i$ ($i = 1, 2, \ldots, 16$) in the form of polynomials, not polynomial equations. Similarly, the same algorithm does not perform factorings. These were added above partially with the help of *Maple V* and partially manually after the derivation of the Gröbner basis $G$.

From the above complete form of $G$, we can now not only check the validity of the compatibility condition (54) for a crack tip (completely equivalently, the condition (36)), but also, if this really holds true, determine further the quantities of interest, i.e. the angle (geometrical quantity) $\theta$ (expressed in $G$ through its cosine and sine, $c_\theta$ and $s_\theta$, respectively) and the mechanical–geometrical quantity $g$. This task can be easily performed on the basis of the first 15 elements of $G$. More explicitly, from $G_{13}$ we find

$$s_\theta^2 = \frac{-\sigma_x\sigma_y + \sigma_y^2 + 2\sigma_{xy}^2}{8(\sigma_x\sigma_y - \sigma_{xy}^2)}, \tag{55}$$

whereas, next, from $G_7$ we obtain

$$c_\theta = \frac{2(1-2s_\theta^2)\sigma_{xy}}{(3\sigma_x - \sigma_y)s_\theta} \tag{56}$$

although we can also use $G_5$ instead (but in this case we obtain two opposite values for $c_\theta$ and this is not recommended). Finally, $G_1$ (a rather obvious equation, resulting simply by adding the first two of Eqs. (33) yields

$$g = \frac{\sigma_x + \sigma_y}{2c_\theta}\,. \tag{57}$$

It is understood that our original system of polynomial equations $P = [P_1, P_2, P_3, P_4]$ is equivalent (as far as its solutions are concerned) to the whole Gröbner basis $G$ and not only to some of its elements, such as $G_1$, $G_7$ and $G_{13}$. Therefore, finally, we have to check the complete original system $P$ with respect to its possible solutions found from Eqs. (55)–(57) unless we prefer to check the remaining elements of $G$ instead (an equivalent possibility).

At this point, it can also be mentioned that it has been impossible, as is clear from Eqs. (33), to exactly determine $s_\theta$ although we have been able to exactly determine $s_\theta^2$ from Eq. (55). Nevertheless, this is not a problem at all since the angle $\theta$ in Eqs. (33) lies in the interval $(-\pi, \pi)$ (for a right crack tip). Then, obviously, the angle $\theta/2$ lies in the interval $(-\pi/2, \pi/2)$ and, therefore, its cosine, $c_\theta$, should be positive. In this way of thinking, we must select in Eq. (56) the correct (the positive) value of $c_\theta$ by using that value (out of two opposite values) of $s_\theta$ which yields a positive value of $c_\theta$.

Under these circumstances, just one point $M_1$ of experimental measurements of the stress components near the crack tip $O$ is sufficient both for the decision about the existence of the crack and for the complete determination of the angle $\theta$. Yet, the angle $\theta$ is not sufficient for the complete location of $O$ with respect to $M_1$ since the distance $r = (OM_1)$ is still unknown. To this end, in case that we are sure about the existence of the crack (validity of the compatibility condition (36), we have to use a second measurement point $M_2$ as well and only in this case the crack tip $O$ will be completely determined (as the cross-section of two straight lines). In such a case, it is also directly possible (without additional measurements) to get also the value of the stress intensity factor $K$ at the crack tip $O$. Now that the position of $O$ with respect to $M_1$ (or, equivalently, to $M_2$) has been determined, we know the distance $r$ and, therefore, $g$, in Eq. (34) directly yields the value of $K$, which is the most important fracture-mechanics quantity at a crack tip.

Finally, it can also be mentioned that in the present application the elements $G_i$ of the Gröbner basis $G$ were not lengthy. Yet, in other elasticity problems (especially when several material/geometrical/loading parameters are used), this will not be the case irrespective of the particular computer implementation of the Buchberger algorithm for the computation of $G$ and this situation should be considered as natural in such a case. The appearance of very lengthy expressions for the elements of $G$ constitutes, of course, a drawback of the present method in complicated cases and two partial remedies are possible: either (i) to reduce the total number of parameters used and/or (ii) change the ordering in the set of variables $V$ (in the purely lexicographic ordering used) so that the easiest to compute parameters can be eliminated first.

It would also be possible to use a mixed ordering of the variables in $V$ in the sense that the stress components be eliminated first, but, next, another, more efficient ordering (such as the total ordering instead of the purely lexicographic ordering) could be used. Such an approach seems not available in popular implementation of the Buchberger algorithm, but even if it were, its practical efficiency would be doubtful beyond the compatibility condition itself since the parameters of interest (in the case of validity of this condition) in general could not be determined from the first elements of the Gröbner basis $G$ (because of our having abandoned purely lexicographic ordering, leading to the "triangularization" of the system of polynomial equations). In any case, if several parameters are unavoidable, we should expect serious difficulties in the computation of $G$ (as is always the case with Gröbner bases) but a good "strategy" in the ordering of these variable in $V$ may be very helpful.

### 3.5. *A three-dimensional application*

As our final application, we consider the three-dimensional singularity due to a concentrated force $F$ applied at the point $O = (0,0,0)$ along the $Oz$-axis inside an infinitely large isotropic elastic medium. This problem (frequently called the Kelvin problem [3, p. 91]) leads to the following formulae for the six stress components [3, pp. 91–93]:

$$\sigma_x = gz[n(x^2+y^2+z^2)-3x^2], \quad \sigma_y = gz[n(x^2+y^2+z^2)-3y^2],$$
$$\sigma_z = -gz[n(x^2+y^2+z^2)+3z^2], \quad \sigma_{xy} = -3gxyz, \tag{58}$$
$$\sigma_{yz} = -gy[n(x^2+y^2+z^2)+3z^2], \quad \sigma_{zx} = -gx[n(x^2+y^2+z^2)+3z^2],$$

where $n$ is the constant

$$n = 1-2\nu, \tag{59}$$

$\nu$ being the Poisson ratio of the isotropic elastic material, and $g$ is a variable defined by

$$g = \frac{F}{8\pi(1-\nu)R^5} \quad \text{with} \quad R^2 = x^2+y^2+z^2. \tag{60}$$

We decided to use all of the above stress components, that is six "hypotheses" in the vocabulary of artificial intelligence and automated theorem proving or, equivalently, three "polynomials" in the vocabulary of Gröbner bases. Working with *Maple V* exactly as in the preceding applications with the list of variables

$$V = [g, x, y, z, \sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}] \tag{61}$$

(leaving $n$ as a simple symbol/parameter) and without options (that is, essentially, with the default option of total degree ordering so that the required computer time can be greatly reduced), we found the related Gröbner basis, which now consists of thirty elements.

The following elements of this Gröbner basis are free from the Cartesian coordinates $(x,y,z)$ and, of course, from the variable $g$ as well (which includes these coordinates because of Eq. (60)) and, therefore, are of interest as compatibility equations for the stress components:

$$\sigma_{yz}[3\sigma_x + n(\sigma_y+\sigma_z)] + (n-3)\sigma_{xy}\sigma_{zx} = 0,$$
$$\sigma_{zx}[3\sigma_y + n(\sigma_x+\sigma_z)] + (n-3)\sigma_{xy}\sigma_{yz} = 0, \tag{62}$$
$$3n(\sigma_x^2+\sigma_y^2) + n^2\sigma_z^2 + (n^2+9)\sigma_x\sigma_y - (n^2-6n+9)\sigma_{xy}^2 + n(n+3)\sigma_z(\sigma_x+\sigma_y) = 0.$$

The above three elements, which can serve as compatibility equations, contain monomials of the second degree with respect to the stress components. Furthermore, the above construction of the present Gröbner basis revealed three more compatibility equations with monomials of the third degree with respect to the same components. These equations are

$$
\begin{gathered}
n[\sigma_z^3 + \sigma_y(\sigma_{zx}^2 + \sigma_{yz}^2) + \sigma_z\sigma_{zx}^2] + (n+3)\sigma_y\sigma_z^2 + (n-3)\sigma_z\sigma_{yz}^2 = 0,\\
n(\sigma_y\sigma_{zx}\sigma_{yz} + \sigma_{zx}^2\sigma_{xy}) + 3\sigma_z^2\sigma_{xy} + (n-3)\sigma_z\sigma_{yz}\sigma_{zx} = 0,\\
3n\sigma_z^3 + 3(n+3)\sigma_x\sigma_z^2 + 3n\sigma_y\sigma_{zx}^2 - n^2\sigma_y\sigma_{yz}^2 + (n^2+3n-9)\sigma_z\sigma_{zx}^2\\
+ n(n+3)\sigma_x\sigma_{zx}^2 - n(n-3)\sigma_z\sigma_{yz}^2 = 0.
\end{gathered}
\tag{63}
$$

Of course, by changing the order of the polynomials and variables in the computer command for the present Gröbner basis, we could obtain additional compatibility equations, but, evidently, these would be dependent on the above six equations as polynomial linear combinations of these equations.

Finally, in a particular experimental problem, having at first verified the validity of all of the above compatibility equations, we can go further and use additional elements of the Gröbner basis (for the sake of space not displayed here), which will easily permit us to determine the origin $O$ of the Cartesian coordinate system (where the load $F$ is applied) as well as the value of this load $F$. Obviously, we assumed above that the Poisson ratio $\nu$ of the isotropic elastic material is a known constant. If it is not, it can be experimentally measured at the points of observation together with the stress components.

## 4. Conclusions–discussion

From the results displayed in the previous section we conclude that it is more or less a simple task to derive purely algebraic polynomial compatibility equations for the stress components in several well-known cases of singularities (either loading or geometrical) both in two- and in three-dimensional media. The approach of Gröbner bases, which has been employed for the execution of the above task, seems to be the best related possibility at least in the applications of the previous section although the alternative elimination methods mentioned in Section 2 are also applicable (but with more care). Of course, beyond deciding about the existence or otherwise of a singularity, the present method permits us to find the exact position of the singularity and its "strength" (e.g. the stress intensity factor in crack problems) if the singularity is really detected. Only two or three observation points, where experimental information must be available, are necessary to complete the above task.

The present approach seems also to be of some theoretical interest mainly because of the compatibility equations derived by the computer. For example, so far we have not seen the fundamental compatibility Eq. (36) in the books and papers on fracture mechanics available to us although it would be reasonable that it would have been derived because of the great importance of crack problems in the recent research in applied mechanics. If this is true, it seems that this equation has not been derived before simply because of the related great computational difficulty without the valuable help of the computer and the related software.

The results of the previous two sections make clear the obvious fact that computer algebra systems (like *Maple V* here) and related algorithms (like Gröbner bases here) frequently offer an excellent computational environment for the derivation of new results, *of course not ideas*, like the compatibility and additional equations in the previous section. Therefore, their use for trivial (as a principle), but lengthy and extremely painful, computations is greatly encouraged. References [24–34] constitute applied/structural mechanics examples of symbolic computations in the computer, but with the present approach of Gröbner bases. Quite recently [44], the author has become able to combine Gröbner bases with computational quantifier elimination algorithms (see e.g. [45]) and investigated the reality and positivity of the solutions to the inverse torsion problem studied in Ref. [29]. (We are not aware of additional applications of Gröbner bases in applied mechanics beyond inverse robot kinematics [6, 13] and mechanisms [35] already mentioned in Section 1.) In any case, with the help of the computer it is possible to find several formulae which we would not attempt to derive by hand although it is still up to us to decide about their real usefulness in practical engineering problems.

Gröbner bases (together with Buchberger's algorithm and its various improvements and modifications) constitute a very efficient tool in computer algebra and its use in applied mechanics can lead to interesting results such as those of the previous section. Yet, the situation is not always ideal: we should use efficient algorithms in complicated problems and take care not to cause too much "pain" to the computer, probably by changing the order of polynomials and variables in an appropriate way, by reducing their number, by appropriately using the options available in the computer software, etc. Otherwise, the computer will complain simply by not deriving the required Gröbner basis even after several hours or even days and, sometimes, by exhausting the available memory of the computer.

Concerning algorithms for Gröbner bases, the *Reduce* updated algorithm in version 3.4 was found faster than the algorithms both of *Maple V* (before its recent release 5) and of *Mathematica* [13, p. 493] although the very recent *Mathematica* implementation of Gröbner bases [19] (not tested by the author yet) may be even more efficient. Therefore, the *Reduce* and *Mathematica* implementations may be preferred in difficult applications. This is just the practical aspect of the problem for the interested engineer.

On the other hand, the theory of the algorithms for the derivation of Gröbner bases is continuously being improved so that they can become more powerful (see e.g. [46]) and their implementation in the commercial software should be always expected. Moreover, further research on Gröbner bases is also continuously taking place (see e.g. [47]) including the suggestion of floating-point Gröbner bases [48] not requiring exact arithmetic in the computer. Similarly, applications of Gröbner bases appear also quite frequently (see e.g. [49]). The interested reader can consult the *Journal of Symbolic Computation* as well as the Proceedings of the ACM ISSAC Conferences on symbolic and algebraic computation (published by ACM, see e.g. [50]) and the papers presented to the recent IMACS ACA Conferences on applications of computer algebra (see e.g. [51]), where not only Gröbner bases, but also competitive polynomial elimination algorithms (and many additional algorithms) are studied and employed in practical problems (unfortunately outside the broad area of applied and structural mechanics). During 1997 both of these Conferences were held (one after the other) in Maui, Hawaii (in July 1997).

The algorithms on Gröbner bases are being constantly improved [46] as well as their computer implementations (e.g. *Maple V* includes such improvements in the Buchberger algorithm in its very recent release: release 5) and similar (or rather even more clear) is the case for their interest and popularity. The appearance of several chapters in books and of excellent devoted books [13, 14] in the field will probably make this approach standard in the future and further related applied/structural mechanics applications are expected to appear. But, on the other hand, the present method for the detection and the location of singularities in elasticity problems beyond its possible theoretical interest (especially in the compatibility equations for the stress or strain components), requires the active help of the techniques in experimental stress analysis and the related progress as well. Unfortunately, this last area falls rather outside the present author's knowledge and the related task for the practical/experimental application of the present hybrid approach should be left to the interested reader.

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## References

[1] Timoshenko, S. P. and Goodier, J. N., *Theory of Elasticity*, 3rd Edn. McGraw-Hill, New York, 1970.

[2] Dally, J. W. and Riley, W. F., *Experimental Stress Analysis*, 3rd Edn. McGraw-Hill, New York, 1991.

[3] Rekach, V. G., *Manual of the Theory of Elasticity*, 1st English Edn. Mir Publishers, Moscow, 1979.

[4] Gröbner, W., *Moderne algebraische Geometrie (Modern Algebraic Geometry)*. Springer, Wien–Innsbruck (1949). http://www.springer.com/de/book/9783211800904

[5] Buchberger, B., Gröbner bases: an algorithmic method in polynomial ideal theory. In: *Multidimensional Systems Theory: Progress, Directions and Open Problems in Multidimensional Systems* (Edited by Bose, N. K.). D. Reidel, Dordrecht, 1985, Chap. 6, pp. 184–232. https://link.springer.com/chapter/10.1007/978-94-017-0275-1_4

[6] Buchberger, B., Applications of Gröbner bases in non-linear computational geometry. In: *Trends in Computer Algebra*, International Symposium, Bad Neuenahr, May 1987, Proceedings (Edited by Janßen, R.). *Lecture Notes in Computer Science*, Vol. 296. Springer, Berlin, 1988, pp. 52–80. https://link.springer.com/chapter/10.1007/3-540-18928-9_5

[7] Davenport, J. H., Siret, Y. and Tournier, E., *Computer Algebra: Systems and Algorithms for Algebraic Computation*, 2nd English Edn. Academic Press, London, 1993. (Originally translated from the French Edn: *Calcul Formel: Systèmes et Algorithmes de Manipulations Algébriques*. Masson, Paris, 1987.) http://staff.bath.ac.uk/masjhd/masternew.pdf

[8] Brackx, F. and Constales, D., *Computer Algebra with LISP and Reduce: an Introduction to Computer-aided Pure Mathematics*. Kluwer, Dordrecht, 1991. http://www.springer.com/us/book/9780792314417

[9] MacCallum, M. A. H. and Wright, F. J., *Algebraic computing with REDUCE*. Lecture Notes from the First Brazilian School on Computer Algebra, Vol. 1 (Edited by Rebouças, M. J. and Roque W. L.). Clarendon Press and Oxford University Press, Oxford, 1991.

[10] Geddes, K. O., Czapor, S. R. and Labahn, G., *Algorithms for Computer Algebra*. Kluwer, Boston, 1992. http://www.springer.com/la/book/9780792392590

[11] Mishra, B., *Algorithmic Algebra*. Springer, New York, 1993. http://www.springer.com/la/book/9780387940908

[12] Winkler, F., *Polynomial Algorithms in Computer Algebra*. Springer, Wien, 1996. http://www.springer.com/us/book/9783211827598

[13] Cox, D., Little, J. and O'Shea, D., *Ideals, Varieties, and Algorithms: an Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer, New York, 1992. http://www.springer.com/cn/book/9781441922571

[14] Becker, T., Weispfenning, V. and Kredel, H., *Gröbner Bases: a Computational Approach to Commutative Algebra*. Springer, New York, 1993. http://www.springer.com/gp/book/9780387979717

[15] Heal, K. M., Hansen, M. L. and Rickard, K. M., *Maple V Learning Guide*. Springer, New York, 1996. http://www.springer.com/gp/book/9780387983974

[16] Monagan, M. B., Geddes, K. O., Heal, K. M., Labahn, G. and Vorkoetter, S., *Maple V Programming Guide*. Springer, New York, 1996. http://www.springer.com/gp/book/9780387983981

[17] Char, B. W., Geddes, K. O., Gonnet, G. H., Leong, B. L., Monagan, M. B. and Watt, S. M., *Maple V Library Reference Manual*. Springer, New York, 1991. http://www.springer.com/us/book/9781475721331

[18] Jenks, R. D. and Sutor, R. S., *Axiom: the Scientific Computation System*. The Numerical Algorithms Group, Oxford, and Springer, New York, 1992. http://www.springer.com/gp/book/9781461277293

[19] Wolfram, S., *The Mathematica Book*, 3rd Edn. Wolfram Media, Champaign, IL, and Cambridge University Press, Cambridge, 1996. *Mathematica:* https://www.wolfram.com/mathematica

[20] Buchberger, B., An implementation of Gröbner bases in *Mathematica*. Technical Report No. 90–58.0, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, 1991. https://pdfs.semanticscholar.org/65ae/e1d19c159d7be657514631ff8f43d890af8f.pdf

[21] Buchberger, B., Collins, G. E, Encarnación, M. J., Hong, H., Johnson, J. R., Krandick, W., Loos, R., Mandache, A. M., Neubacher, A. and Vielhaber, H., SACLIB 1.1 user's guide. Technical Report No. 93–19, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, 1993.

[22] Windsteiger, W. and Buchberger, B., GRÖBNER: a library for computing Gröbner bases based on SACLIB. Technical Report No. 93–72, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, 1993. https://www.researchgate.net/profile/Bruno_Buchberger/publication/242609212_Groebner_a_library_for_computing_grobner_bases_based_on_saclib/links/00b49520366f04b75e000000.pdf

[23] Windsteiger, W., Using GRÖBNER as a "black box". Technical Report No. 93–71, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, 1993. https://pdfs.semanticscholar.org/2c8d/338cf45330a857d56bbc28710bb61a63125c.pdf

[24] Ioakimidis, N. I. and Anastasselou, E. G., Application of Gröbner bases to problems of movement of a particle. *Computers and Mathematics with Applications*, **27**, 51–57 (1994). https://doi.org/10.1016/0898-1221(94)90046-9

[25] Ioakimidis, N. I. and Anastasselou, E. G., Computer-based manipulation of systems of equations in elasticity problems with Gröbner bases. *Computer Methods in Applied Mechanics and Engineering*, **110**, 103–111 (1993). https://doi.org/10.1016/0045-7825(93)90022-P

[26] Anastasselou, E. G., Determination of the loading at infinity from the stress intensity factors with computer algebra software. *International Journal of Fracture*, **63**, R35–R39 (1993). https://doi.org/10.1007/BF00017289

[27] Ioakimidis, N. I. and Anastasselos, G. T., On the evaluation of stress intensity factors for a simple crack under parametric loading. *Computers and Structures*, **51**, 791–794 (1994). https://doi.org/10.1016/S0045-7949(05)80019-7

[28] Ioakimidis, N. I. and Anastasselou, E. G., Gröbner bases in truss problems with *Maple*. *Computers and Structures*, **52**, 1093–1096 (1994). https://doi.org/10.1016/0045-7949(94)90093-0

[29] Ioakimidis, N. I., Symbolic computations for the solution of inverse/design problems with *Maple*. *Computers and Structures*, **53**, 63–68 (1994). https://doi.org/10.1016/0045-7949(94)90130-9

[30] Ioakimidis, N. I. and Anastasselou, E. G., Derivation of the equation of caustics in Cartesian coordinates with *Maple*. *Engineering Fracture Mechanics*, **48**, 147–149 (1994). https://doi.org/10.1016/0013-7944(94)90151-1

[31] Ioakimidis, N. I., Computer-generated formulae for the location of straight cracks. *Engineering Fracture Mechanics*, **51**, 847–850 (1995). https://doi.org/10.1016/0013-7944(94)00324-B

[32] Ioakimidis, N. I., Determination of critical buckling loads with Gröbner bases. *Computers and Structures*, **55**, 433–440 (1995). https://doi.org/10.1016/0045-7949(95)98869-R

[33] Ioakimidis, N. I. and Anastasselou, E. G., Annihilation of loading parameters in classical numerical methods with differential equations. *Computers and Structures*, **59**, 265–271 (1996). https://doi.org/10.1016/0045-7949(95)00257-X

[34] Ioakimidis, N. I., The equation of caustics in Cartesian coordinates for mixed-mode stress intensity factors. *Engineering Fracture Mechanics*, **54**, 441–443 (1996). https://doi.org/10.1016/0013-7944(95)00208-1

[35] Kalker–Kalkman, C. M., An implementation of Buchberger's algorithm with applications to robotics. *Mechanism and Machine Theory*, **28**, 523–537 (1993). https://doi.org/10.1016/0094-114X(93)90033-R

[36] Kapur, D. and Lakshman, Y. N., Elimination methods: an introduction. In: *Symbolic and Numerical Computation for Artificial Intelligence* (Edited by Donald, B. R., Kapur, D. and Mundy, J. L.). Academic Press, London, 1992, Chap. 2, pp. 45–87. https://www2.cs.duke.edu/donaldlab/Books/SymbolicNumericalComputation/045-087.pdf

[37] Ritt, J. F., *Differential Algebra*. Dover, New York, 1966. (Originally published by the American Mathematical Society, Providence, RI, 1950.) http://www4.ncsu.edu/ singer/foraachen/DA.pdf

[38] Wu, W.-t, *Mechanical Theorem Proving in Geometries: Basic Principles*. Springer, Wien, 1994. (Translated from the original Chinese Edn: *Basic Principles of Mechanical Theorem Proving in Geometries*. Science Press, Beijing, 1984.) http://www.springer.com/us/book/9783211825068

[39] Wang, D., Reasoning about geometric problems using an elimination method. In: *Automated Practical Reasoning: Algebraic Approaches* (Edited by Pfalzgraf, J. and Wang, D.). Springer, Wien, 1995, pp. 147–185. https://link.springer.com/chapter/10.1007/978-3-7091-6604-8_8

[40] Wang, D., An implementation of the characteristic set method in *Maple*. In: *Automated Practical Reasoning: Algebraic Approaches* (Edited by Pfalzgraf, J. and Wang, D.). Springer, Wien, 1995, pp. 187–201. https://link.springer.com/chapter/10.1007/978-3-7091-6604-8_9

[41] Anastasselou, E. G., Compatibility differential/difference equations in classical beam problems with the characteristic sets method. *Advances in Engineering Software*, **22**, 63–68 (1995). https://doi.org/10.1016/0965-9978(95)00002-E

[42] Aliabadi, M. H. and Rooke, D. P., *Numerical Fracture Mechanics*. Computational Mechanics Publications, Southampton, and Kluwer, Dordrecht, 1991. http://www.springer.com/la/book/9780792311751

[43] Hills, D. A., Kelly, P. A., Dai, D. N. and Korsunsky, A. M., *Solution of Crack Problems: the Distributed Dislocation Technique*. Kluwer, Dordrecht, 1996. http://www.springer.com/gp/book/9780792338482

[44] Ioakimidis, N. I., Solvability conditions in simple inverse/design torsion problems, 1999 (in preparation).

[45] Caviness, B. F. and Johnson, J. R. (Eds.), *Quantifier Elimination and Cylindrical Algebraic Decomposition*. Springer, Wien, 1998. http://www.springer.com/us/book/9783211827949

[46] Czapor, S. R., A heuristic selection strategy for lexicographic Gröbner bases? In: *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation* (ISSAC '91), July 15–17, 1991, Bonn (Edited by Watt, S. M.). ACM Press, New York, 1991, pp. 39–48. https://dl.acm.org/citation.cfm?id=120700

[47] Pauer, F. and Zampieri, S., Gröbner bases with respect to generalized term orders and their application to the modelling problem. *Journal of Symbolic Computation*, **21**, 155–168 (1996). https://doi.org/10.1006/jsco.1996.0007

[48] Shirayanagi, K., An algorithm to compute floating point Groebner bases. In: *Mathematical Computation with Maple V: Ideas and Applications*, Proceedings of the *Maple* Summer Workshop and Symposium, University of Michigan, Ann Arbor, June 1993 (Edited by Lee, T.). Birkhäuser, Boston, 1993, pp. 95–106. https://link.springer.com/chapter/10.1007/978-1-4612-0351-3_10

[49] Alwash, M. A. M., Periodic solutions of a quartic differential equation and Groebner bases. *Journal of Computational and Applied Mathematics*, **75**, 67–76 (1996). https://doi.org/10.1016/S0377-0427(96)00059-3

[50] Lakshman, Y. N. (Ed.), *Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation* (ISSAC '96), ETH, Zurich, July 1996. ACM Press, New York, 1996. https://dl.acm.org/citation.cfm?id=236869

[51] Wester, M., Steinberg, S. and Jahn, M. (Eds.), *Electronic Proceedings of the 1st International IMACS Conference on Applications of Computer Algebra*, University of New Mexico, Albuquerque, May 1995. http://math.unm.edu/ACA/1995/Proceedings/MainPage.html