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A new approach to the construction of some path-independent integrals about crack tips

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Abstract A new method, based on the complex potentials of Kolosov–Muskhelishvili and Cauchy's theorem in complex analysis, is applied to the establishment of path-independence of some integrals along a curve surrounding the tip of a crack in plane isotropic or anisotropic elasticity. The cases considered are (i) of loaded straight cracks in plane isotropic elasticity, (ii) of unloaded cracks having the shape of a circular arc (circular-arc-shaped cracks) in plane isotropic elasticity and (iii) of unloaded straight cracks in plane anisotropic elasticity. Further generalizations of the proposed method can be easily made.

Keywords Path-independent integrals · Plane isotropic elasticity · Plane anisotropic elasticity · Fracture mechanics · Straight cracks · Circular-arc-shaped cracks · Complex potentials of Kolosov–Muskhelishvili · Holomorphic functions · Cauchy theorem · Boundary conditions

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1. Introduction

Path-independent integrals were proved very useful in plane isotropic elasticity of cracked media. Significant contributions to these integrals are due to Eshelby (see, e.g., [1]) and to Rice (see, e.g., [2]). Many applications of this class of integrals have appeared in a lot of papers. The corresponding literature is both too extensive and very well known and it will not be reported here. In this paper, we will consider a new method for the construction of some of these integrals and the proof of their path-independence about crack tips. This method is an extension of previous results [3–8] on complex path-independent integrals about crack tips and is based on the use of the well-known complex potentials of Kolosov–Muskhelishvili [9], the Cauchy theorem of complex analysis [10] and, particularly, the use of the boundary conditions along the crack edges in complex form through the complex potentials of Kolosov–Muskhelishvili. The result that we obtain in the special case of the classical J-integral coincides with that derived by Budiansky and Rice [11] by another method, but by making also use of the complex potentials of Kolosov–Muskhelishvili. One application of the expression of the J-integral in terms of the aforementioned complex potentials is contained in [12].

The present results can easily be generalized to various directions. The cases of circular-arc-shaped cracks and of straight cracks inside plane anisotropic media are used here as simple applications of the proposed technique. Further applications of the present results can also be made quite straightforwardly.

2. Loaded straight crack in a plane isotropic medium

At first, we consider a smooth curvilinear crack L inside a plane isotropic elastic medium under plane strain or generalized plane stress conditions. We denote by $+$ and $-$ the two edges of the crack and by $t = x + iy$ the points of the crack (Fig. 1), where x and y are the Cartesian coordinates in the cracked finite or infinite plane elastic medium. We also denote by z the points $x + iy$ of the plane and we assume known the loading functions $\sigma_n^{\pm}(t)$ and $\sigma_t^{\pm}(t)$ (normal and tangential loading distributions, respectively, along the crack edges and with the usual sign conventions). Under these assumptions, the complex potentials $\Phi(z)$ and $\Psi(z)$ of Kolosov–Muskhelishvili [9] have to satisfy along the crack the following boundary conditions [9, 13]

$$\Phi^{\pm}(t) + \overline{\Phi^{\pm}(t)} + \frac{\mathrm{d}t}{\overline{\mathrm{d}t}}\left[\bar{t}\Phi'^{\pm}(t) + \Psi^{\pm}(t)\right] = P^{\pm}(t) \tag{1}$$

with

$$P^{\pm}(t) = \sigma_n^{\pm}(t) - i\sigma_t^{\pm}(t) \tag{2}$$

and where a bar denotes the complex conjugate of a complex number or a complex function and the symbol $\mathrm{d}t/\overline{\mathrm{d}t}$ is defined as

$$\frac{\mathrm{d}t}{\overline{\mathrm{d}t}} = \frac{\mathrm{d}t/\mathrm{d}s}{\overline{\mathrm{d}t}/\mathrm{d}s} \tag{3}$$

with s denoting the arc-length along the crack. Finally, it should be emphasized that $\Phi(z)$ and $\Psi(z)$ are complex functions holomorphic in the plane elastic medium with the exception of the points of

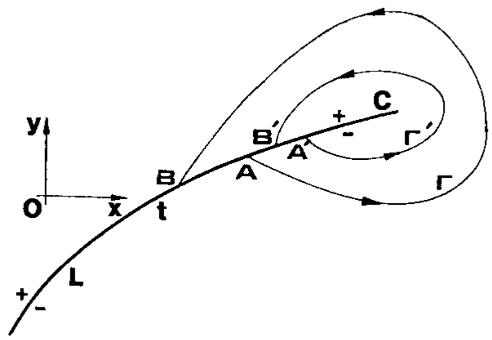

Figure 1: Geometry of the crack and of the integration paths.

the crack L. On the contrary, $\sigma_n^{\pm}(t)$ and $\sigma_t^{\pm}(t)$ simply denote the loading distributions along the crack edges and are not related at all to holomorphic functions.

In this section, we will restrict ourselves to straight crack problems. We also assume that $P^{\pm}(t)$ are the boundary values along the crack edges of a complex function $P(z)$ holomorphic in the elastic medium under consideration with the exception of the crack L. This happens, for example, if $P^{\pm}(t)$ vanish or reduce to a constant or to a simple polynomial, but more complicated cases are also convenient several times. Moreover, we assume that the crack lies along the Ox-axis. We also consider two arbitrary curves Γ and Γ' (Fig. 1) with ends on the lower and upper edges of the crack (A and B for Γ, and A' and B' for Γ', Fig. 1). We will construct the classical J-integral by a new (to this author's best knowledge) method.

Under the above assumptions, we rewrite Eq. (1) as

$$\Phi^{\pm}(t) + \overline{\Phi^{\pm}(t)} + t\Phi'^{\pm}(t) + \Psi^{\pm}(t) = P^{\pm}(t). \tag{4}$$

By multiplying Eq. (4) by $2\Phi^{\pm}(t)$, we obtain

$$[\Phi^{\pm}(t)]^2 + 2\Phi^{\pm}(t)\Psi^{\pm}(t) + \left\{t[\Phi^{\pm}(t)]^2\right\}' + 2|\Phi^{\pm}(t)|^2 = 2\Phi^{\pm}(t)P^{\pm}(t) \tag{5}$$

and, since the last term of the left-hand side of Eq. (5) is a real quantity,

$$\operatorname{Im}\left\{[\Phi^{\pm}(t)]^2 + 2\Phi^{\pm}(t)\Psi^{\pm}(t) - 2\Phi^{\pm}(t)P^{\pm}(t) + \left[t[\Phi^{\pm}(t)]^2\right]'\right\} = 0. \tag{6}$$

Since Eq. (6) holds true along both crack edges, it is further evident for the complex function

$$F(z) = \Phi^2(z) + 2\Phi(z)\Psi(z) - 2\Phi(z)P(z) + \left[z\Phi^2(z)\right]', \tag{7}$$

which satisfies the Cauchy theorem [10]

$$\int_{\gamma} F(z)\,\mathrm{d}z = 0 \tag{8}$$

along a contour γ inside which $F(z)$ is holomorphic, that

$$\operatorname{Im}\int_{\Gamma} F(z)\,\mathrm{d}z = \operatorname{Im}\int_{\Gamma'} F(z)\,\mathrm{d}z, \tag{9}$$

where Γ and Γ' were already defined (Fig. 1). From Eq. (9) we conclude the path-independence of the quantity

$$K_1 = \operatorname{Im}\left\{\int_A^B \left[\Phi(z)[\Phi(z) + 2\Psi(z) - 2P(z)]\right]\mathrm{d}z + \left[z\Phi^2(z)\right]_A^B\right\} \tag{10}$$

with the integral considered along a curve Γ surrounding the crack tip C (Fig. 1). This is exactly the Rice classical J-integral (and really it can easily be written as a single integral) for this case, but multiplied by $E/2$ with E denoting the modulus of elasticity in tension of the material of the elastic medium. This is clear if the results of Budiansky and Rice [11] are taken into consideration.

At this point, we wish to emphasize that it was the boundary condition (6) which was the initial point for the construction of the path-independent integral K_1. An inappropriate selection of $F(z)$ in Eq. (6) would lead to contributions in Eq. (8) from the parts of the crack edges between A and A' as well as B and B' (Fig. 1) with γ taken as the union of Γ (in the clockwise direction) and Γ' (in the anticlockwise direction) and the aforementioned parts of the crack edges so that the contour γ not surrounding the crack tip C can be formed.

Concluding this section, we wish to mention that in the special case when $P(z) \equiv 0$, that is the crack is unloaded, the path-independent integral K_1 in Eq. (10) takes the following simpler form:

$$K_2 = \mathrm{Im}\Big\{ \int_A^B \big[\Phi(z)[\Phi(z) + 2\Psi(z)]\big]\,\mathrm{d}z + \big[z\Phi^2(z)\big]_A^B \Big\}. \tag{11}$$

This quantity, K_2, is exactly the classical J-integral, but multiplied again by $E/2$. This fact was already realized by Theocaris and Tsamasphyros [5–7] (on the basis of the results of Budiansky and Rice [11]) by using a method somewhat different from that used here, but also based on complex path-independent integrals.

3. Circular-arc-shaped crack

Now we will generalize the results of Section 2 to the case when the crack L has the shape of an arc of a circle (circular-arc-shaped crack). We assume this crack unloaded (the case when it is loaded can be considered as was already made in the previous section for a straight crack) and, furthermore, for convenience and without loss of generality, that the radius of the circle is equal to 1 and that its centre coincides with the origin O of the Cartesian coordinate system Oxy (Fig. 1).

Under these conditions, we have for the points t of the crack

$$t\bar{t} = 1, \quad \bar{t} = \frac{1}{t}, \quad \overline{\mathrm{d}t} = -\frac{\mathrm{d}t}{t^2}, \quad \frac{\mathrm{d}t}{\overline{\mathrm{d}t}} = -t^2 \tag{12}$$

and the fundamental boundary conditions (1) along the crack edges take the form (for an unloaded crack)

$$\Phi^{\pm}(t) + \overline{\Phi^{\pm}(t)} - t^2[t^{-1}\Phi'^{\pm}(t) + \Psi^{\pm}(t)] = 0. \tag{13}$$

By multiplying this equation by $2\Phi^{\pm}(t)$ (exactly as in Section 2), we obtain

$$3[\Phi^{\pm}(t)]^2 - 2t^2\Phi^{\pm}(t)\Psi^{\pm}(t) - \big\{t[\Phi^{\pm}(t)]^2\big\}' + 2|\Phi^{\pm}(t)|^2 = 0 \tag{14}$$

and, furthermore, that

$$\mathrm{Im}\Big\{3[\Phi^{\pm}(t)]^2 - 2t^2\Phi^{\pm}(t)\Psi^{\pm}(t) - \big\{t[\Phi^{\pm}(t)]^2\big\}'\Big\} = 0. \tag{15}$$

Hence, in the present case, the appropriate function $F(z)$ is given by

$$F(z) = 3\Phi^2(z) - 2z^2\Phi(z)\Psi(z) - \big[z\Phi^2(z)\big]' \tag{16}$$

and the corresponding path-independent quantity (analogous to K_1 and K_2 for the cases considered previously) is

$$K_3 = \mathrm{Im}\Big\{ \int_A^B \big[\Phi(z)[3\Phi(z) - 2z^2\Psi(z)]\big]\,\mathrm{d}z - \big[z\Phi^2(z)\big]_A^B \Big\}. \tag{17}$$

4. Straight crack in a plane anisotropic medium

As a final application of the present technique already suggested in Section 2, we consider an anisotropic elastic medium under plane conditions with the plane being a plane of elastic symmetry. In this case, the stress components in the medium are expressible in terms of two complex potentials, $\Phi(z_1)$ and $\Psi(z_2)$, [14], where $z_{1,2} = x + \mu_{1,2}\, y$ with Oxy a Cartesian coordinate system (with axes of appropriate orientation) and $\mu_{1,2}$ appropriate complex constants of the material of the anisotropic elastic medium (with $\mu_1 \neq \mu_2$) [14]. Now we assume now the existence of a simple straight unloaded crack along the Ox-axis. (The case of a loaded crack can in general be considered as well in a manner analogous to that used in Section 3). Under these circumstances, the following boundary conditions can be seen to hold on the basis of the results of References [13, 15]

$$(\mu_1 - \bar{\mu}_2)\Phi^{\pm}(t) + (\bar{\mu}_1 - \bar{\mu}_2)\overline{\Phi^{\pm}(t)} + (\mu_2 - \bar{\mu}_2)\Psi^{\pm}(t) = 0. \tag{18}$$

These conditions can also be written as

$$\alpha\Phi^{\pm}(t) + \overline{\Phi^{\pm}(t)} + \beta\Psi^{\pm}(t) = 0 \tag{19}$$

with

$$\alpha = \frac{\mu_1 - \bar{\mu}_2}{\bar{\mu}_1 - \bar{\mu}_2} \quad \text{and} \quad \beta = \frac{\mu_2 - \bar{\mu}_2}{\bar{\mu}_1 - \bar{\mu}_2}. \tag{20}$$

Now we proceed to the construction of a path-independent integral along a curve Γ surrounding the crack tip C (Fig. 1). To this end, we multiply Eq. (19) by $\Phi^{\pm}(t)$ and we directly obtain

$$\alpha[\Phi^{\pm}(t)]^2 + \beta\Phi^{\pm}(t)\Psi^{\pm}(t) + |\Phi^{\pm}(t)|^2 = 0 \tag{21}$$

or, further,

$$\operatorname{Im}\left\{\alpha[\Phi^{\pm}(t)]^2 + \beta\Phi^{\pm}(t)\Psi^{\pm}(t)\right\} = 0. \tag{22}$$

Next, we define the complex function

$$F(z) = \alpha\Phi^2(z) + \beta\Phi(z)\Psi(z), \tag{23}$$

which satisfies the Cauchy theorem (8) on a contour γ inside which $F(z)$ is holomorphic. Then we easily see that the integral

$$K_4 = \operatorname{Im}\int_A^B \{\Phi(z)[\alpha\Phi(z) + \beta\Psi(z)]\}\,\mathrm{d}z \tag{24}$$

is independent of the curve Γ surrounding the crack tip C (Fig. 1) .

One remark of major importance has to be made here. The stress and strain components at a point (x,y) of a plane anisotropic elastic medium depend on the complex potentials $\Phi(z)$ and $\Psi(z)$ but with $z = z_1 = x + \mu_1\, y$ and $z = z_2 = x + \mu_2\, y$, respectively. Hence, in Eq. (24) we work essentially along two curves, Γ_1 and Γ_2, corresponding to Γ. The points of Γ_1 and Γ_2 result from those of Γ by dividing y by $-i\mu_1$ and $-i\mu_2$, respectively, and leaving x unchanged. In this way, the complex potential $\Phi(z)$ takes at a point $x + (iy/\mu_1)$ of Γ_1 the value $\Phi[x + \mu_1(iy/\mu_1)] = \Phi(x+iy) = \Phi(z)$, which is really required in Eq. (24). Similarly, $\Psi(z)$ takes at a point $x + (iy/\mu_2)$ of Γ_2 the value $\Psi[x + \mu_2(iy/\mu_2)] = \Psi(x+iy) = \Psi(z)$, which is also really required in Eq. (24). To conclude, Eq. (24) is correct as it stands and with the integration from A to B made along the curve Γ (Fig. 1), but the values $\Phi(z)$ and $\Psi(z)$ at the points of Γ are in reality useful for the evaluation of the stress and strain components at the corresponding points of Γ_1 and Γ_2, respectively. This remark is essentially irrelevant to Eq. (24) itself, but if proper attention is not paid to it, serious mistakes may be made during the evaluation of K_4.

5. Discussion

The results of the previous sections show the usefulness of the complex potentials of plane isotropic and anisotropic elasticity and of the Cauchy theorem in complex analysis in the construction of path-independent integrals about crack tips with the help of the boundary conditions, expressed in terms of the aforementioned complex potentials. Further generalizations and applications of the previous results to more complicated geometry and loading conditions, both in isotropic and in anisotropic elasticity, can easily be made in several cases probable to appear in practice.

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