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On the boundary integral equation method for the problem of a plane crack inside a three-dimensional elastic medium

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Abstract The problem of a plane crack of arbitrary shape and under an arbitrary normal pressure distribution inside an infinite three-dimensional isotropic elastic medium is reconsidered by the boundary integral equation method. This method is seen to be capable to produce the singular integral equation of this problem with one unknown function, i.e. the displacements of the points of the crack faces (and not the derivatives of this function), and this is achieved in two ways. This is an alternative and probably interesting new method for the derivation of the aforementioned singular integral equation having been previously derived by two other methods. Generalizations of the present results to more complicated problems follow trivially.

Keywords Boundary integral equations · Crack problems · Fracture mechanics · Three-dimensional elasticity · Isotropic elasticity · Principal value integrals · Cauchy-type integrals · Singular integral equations · Finite-part integrals · Hypersingular integrals · Hypersingular integral equations

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1. Introduction

The problem of a plane crack of arbitrary shape on the Ox_1x_2-plane inside an infinite three-dimensional isotropic elastic medium and under an arbitrary normal pressure distribution $p(x_1,x_2)$ was recently reconsidered by the author and reduced to the following singular integral equation (or, perhaps better, hypersingular integral equation) on on the crack surface S [1, 2]:

$$\neq\!\!\!\!\!\!\iint_S \frac{f(\xi_1,\xi_2)}{r^3}\,\mathrm{d}S = -\frac{4\pi(1-\nu^2)}{E}\,p(x_1,x_2), \quad (x_1,x_2)\in S, \tag{1}$$

where (ξ_1,ξ_2) denote, like (x_1,x_2), the Cartesian coordinates of the crack faces, $f(\xi_1,\xi_2)$ is the unknown function equal to the absolute value of the displacement of the points (ξ_1,ξ_2) of both crack faces along the Ox_3-axis, r is the distance

$$r = \sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}, \tag{2}$$

E is the Young modulus (modulus of elasticity) and ν the Poisson ratio of the isotropic elastic material.

Moreover, we interpret the integral in Eq. (1) as a finite-part integral in the following sense. If we define the polar coordinates (r,θ) having as centre the point (x_1,x_2) by

$$(\xi_1-x_1)+i(\xi_2-x_2) = re^{i\theta}, \tag{3}$$

then, since

$$\mathrm{d}S = \mathrm{d}\xi_1\,\mathrm{d}\xi_2 = r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{4}$$

the integral in Eq. (1) is defined by

$$\neq\!\!\!\!\!\!\iint_S \frac{f(\xi_1,\xi_2)}{r^3}\,\mathrm{d}S = \int_0^{2\pi}\Big[\,\neq\!\!\!\!\!\!\int_0^{R(\theta)} \frac{f(\xi_1,\xi_2)}{r^2}\,\mathrm{d}r\Big]\mathrm{d}\theta, \tag{5}$$

where the inner integral in Eq. (5) is defined as a one-dimensional finite-part integral. The theory of these integrals and methods for their numerical evaluation can be found, for example, in the work of Kutt [3]. Moreover, $R(\theta)$ in Eq. (5) denotes the maximum value of the polar radius r inside the crack S along the direction θ $(0\le\theta<2\pi)$.

We can add that the right-hand side integral in Eq. (5) can also be written in the form of a Cauchy-type principal value integral, that is [1]

$$\int_0^{2\pi}\Big[\,\neq\!\!\!\!\!\!\int_0^{R(\theta)} \frac{f(\xi_1,\xi_2)}{r^2}\,\mathrm{d}r\Big]\mathrm{d}\theta = \int_0^{\pi}\Big[\frac{\mathrm{d}}{\mathrm{d}s}\,\fint_{-R(-\theta)}^{R(\theta)} \frac{f(\xi_1,\xi_2)}{r-s}\,\mathrm{d}r\Big]_{s=0}\mathrm{d}\theta. \tag{6}$$

The developments of Refs. [1, 2] leading to Eq. (1) were based on the results of Bueckner [4], who derived, by using the Boussinesq–Papkovich potential, the singular integral equation

$$\Delta\iint_S \frac{f(\xi_1,\xi_2)}{r}\,\mathrm{d}S = -\frac{4\pi(1-\nu^2)}{E}\,p(x_1,x_2), \quad (x_1,x_2)\in S \tag{7}$$

(with $\Delta\equiv\nabla^2$) completely equivalent to Eq. (1) [2].

Here for the derivation of Eq. (1) we will use the boundary integral equation method of Cruse [5], which was applied to crack problems by Weaver [6]. Moreover, it will be seen that the singular integral equation of Weaver [6] for our problem,

$$\iint_S \frac{1}{r^2}\Big[\cos\theta\,\frac{\partial f(\xi_1,\xi_2)}{\partial\xi_1}+\sin\theta\,\frac{\partial f(\xi_1,\xi_2)}{\partial\xi_2}\Big]\mathrm{d}S=-\frac{4\pi(1-\nu^2)}{E}\,p(x_1,x_2),\quad (x_1,x_2)\in S, \tag{8}$$

is equivalent to Eq. (1). The same equation, Eq. (8), was also derived by Bui [7].

As regards the question "why to prefer Eq. (1) to Eqs. (7) and (8) since all these three singular integral equations are equivalent," the reply is "simply because Eq. (1) does not contain derivatives, contrary to Eqs. (7) and (8)." Differentiating, contrary to integrating, is generally an undesirable task during numerical computations.

2. Derivation of the singular integral equation

At first, we will use the singular integral equation (8), having been derived in Ref. [6] by the boundary integral equation method, and we will transform it to the singular integral equation (1). To this end, we take into account Eq. (3), which gives

$$\xi_1=x_1+r\cos\theta,\quad \xi_2=x_2+r\sin\theta \tag{9}$$

and we easily find that

$$\begin{aligned}\frac{\partial f}{\partial\xi_1}&=\frac{\partial f}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial f}{\partial\theta}\sin\theta,\\ \frac{\partial f}{\partial\xi_2}&=\frac{\partial f}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial f}{\partial\theta}\cos\theta.\end{aligned} \tag{10}$$

These values of the partial derivatives of f, when inserted in Eq. (8), give

$$\int_0^{2\pi}\Big[\,⨎_0^{R(\theta)}\frac{1}{r}\frac{\partial f}{\partial r}\,\mathrm{d}r\Big]\mathrm{d}\theta=-\frac{4\pi(1-\nu^2)}{E}\,p(x_1,x_2),\quad (x_1,x_2)\in S, \tag{11}$$

where Eq. (4) and the results of Ref. [8] have been taken into consideration. This singular integral equation, Eq. (11), although formally containing just one unknown function, $\partial f/\partial r$, is not convenient in practice since the centre (x_1,x_2) of the polar coordinates (r,θ) is not a constant point but it moves on the whole surface S of the crack.

Now, we rewrite the left-hand side of Eq. (11) as

$$\begin{aligned}\int_0^{2\pi}\Big[\,⨎_0^{R(\theta)}\frac{1}{r}\frac{\partial f}{\partial r}\,\mathrm{d}r\Big]\mathrm{d}\theta&=\int_0^{2\pi}\Big\{\int_0^{R(\theta)}\frac{1}{r}\Big[\frac{\partial f}{\partial r}-\Big(\frac{\partial f}{\partial r}\Big)_{r=0}\Big]\mathrm{d}r\Big\}\mathrm{d}\theta\\&\quad+\int_0^{2\pi}\Big(\frac{\partial f}{\partial r}\Big)_{r=0}\Big[\,⨎_0^{R(\theta)}\frac{1}{r}\,\mathrm{d}r\Big]\mathrm{d}\theta.\end{aligned} \tag{12}$$

We can also mention that $(\partial f/\partial r)_{r=0}$ is a function not only of the point (x_1,x_2) in Eq. (9), but also of the polar angle θ. From the geometry of the crack S it is also clear that

$$\Big(\frac{\partial f}{\partial r}\Big)_{(r=0,\,\theta+\pi)}=-\Big(\frac{\partial f}{\partial r}\Big)_{(r=0,\,\theta)}. \tag{13}$$

Next, we perform an integration by parts on the inner integral of the right-hand side of Eq. (12) and we obtain

$$\int_0^{R(\theta)} \frac{1}{r}\left[\frac{\partial f}{\partial r} - \left(\frac{\partial f}{\partial r}\right)_{r=0}\right] \mathrm{d}r = \frac{f(r,\theta) - f(0,\theta) - r(\partial f/\partial r)(0,\theta)}{r}\Big|_0^{R(\theta)}$$
$$+ \int_0^{R(\theta)} \frac{f(r,\theta) - f(0,\theta) - r(\partial f/\partial r)(0,\theta)}{r^2}\,\mathrm{d}r = \frac{f(R(\theta),\theta) - f(0,\theta)}{R(\theta)}$$
$$- \frac{\partial f}{\partial r}(0,\theta) + \fint_0^{R(\theta)} \frac{f(r,\theta)}{r^2}\,\mathrm{d}r - f(0,\theta)\fint_0^{R(\theta)} \frac{\mathrm{d}r}{r^2} - \frac{\partial f}{\partial r}(0,\theta)\fint_0^{R(\theta)} \frac{\mathrm{d}r}{r}, \qquad (14)$$

where we retained the symbol f even when the function $f(\xi_1,\xi_2)$ is considered as a function of x_1, x_2, r and θ. But it is well known that [3]

$$\fint_0^{R(\theta)} \frac{\mathrm{d}r}{r^2} = -\frac{1}{r}\Big|_{r=R(\theta)} = -\frac{1}{R(\theta)} \qquad (15)$$

and, incidentally, it can also be mentioned that

$$\fint_0^{R(\theta)} \frac{\mathrm{d}r}{r} = \ln r\Big|_{r=R(\theta)} = \ln R(\theta). \qquad (16)$$

By taking into account Eqs. (12), (14) and (15), we conclude that

$$\int_0^{2\pi}\left[\fint_0^{R(\theta)} \frac{1}{r}\frac{\partial f}{\partial r}\,\mathrm{d}r\right]\mathrm{d}\theta = \int_0^{2\pi}\left[\fint_0^{R(\theta)} \frac{f(r,\theta)}{r^2}\,\mathrm{d}r\right]\mathrm{d}\theta - \int_0^{2\pi} \frac{\partial f}{\partial r}(0,\theta)\,\mathrm{d}\theta, \qquad (17)$$

where the obvious fact (clear from the physical meaning of $f(r,\theta)$) that

$$f(R(\theta),\theta) = 0, \quad 0 \le \theta < 2\pi, \qquad (18)$$

was also taken into account. Finally, by taking into account Eq. (13), we understand that the second integral in the right-hand side of Eq. (17) vanishes. This completes the proof of Eq. (1) on the basis of Eq. (8) with Eq. (5) having also been taken into consideration.

The singular integral equation (1) was derived by using the method of Bueckner [4] in Refs. [1, 2] and here by using the singular integral equation (8) of Weaver [6]. Now let us proceed to the direct use of the boundary integral equation method [6] for the derivation of Eq. (1) without resorting to Eq. (8).

At first, for the displacements $u_i(\mathbf{X})$ of the points $\mathbf{X} = (x_1,x_2,x_3)$ of the elastic medium we have [6]

$$u_i(\mathbf{X}) = -\iint_S \Delta u_j(\boldsymbol{\xi})\,T_{ij}^{+}(\boldsymbol{\xi},\mathbf{X})\,\mathrm{d}S, \qquad (19)$$

where $\boldsymbol{\xi} = (\xi_1,\xi_2)$ and $T_{ij}^{+}(\boldsymbol{\xi},\mathbf{X})$ is given by [6]

$$T_{ij}^{+}(\boldsymbol{\xi},\mathbf{X}) = -\frac{K}{R^2}\left[\frac{x_3}{R}\left(\delta_{ij} + \frac{3R_{,i}R_{,j}}{1-2\nu}\right) + \delta_{j3}R_{,i} - \delta_{i3}R_{,j}\right], \qquad (20)$$

where

$$K = \frac{1-2\nu}{8\pi(1-\nu)}, \quad R = |\boldsymbol{\xi} - \mathbf{X}|, \quad R_{,i} = \frac{\partial R}{\partial \xi_i} = \frac{\xi_i - x_i}{R} \qquad (21)$$

and δ_{ij} is the Kronecker delta. Moreover, $\Delta u_j(\boldsymbol{\xi})$ are the differences $u_j^+(\boldsymbol{\xi}) - u_j^-(\boldsymbol{\xi})$ of the displacements of the corresponding points of the two crack faces. Evidently, because of the present restriction of the loading distribution concerning a normal loading $p(x_1, x_2)$ only, it is clear that

$$\Delta u_\alpha(\boldsymbol{\xi}) = 0, \quad \alpha = 1,2 \quad \text{and} \quad \Delta u_3(\boldsymbol{\xi}) = 2f(\boldsymbol{\xi}), \tag{22}$$

where $f(\boldsymbol{\xi})$ is the function already used in Eqs. (1), (8), etc.

In our case, we have to consider only the equation [6]

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu(u_{1,1}^+ + u_{2,2}^+) + (1-\nu)u_{3,3}^+\right] \tag{23}$$

on the crack faces with the boundary condition

$$\sigma_{33}(\mathbf{X}) = -p(\mathbf{X}), \quad \mathbf{X} \in S. \tag{24}$$

We will evaluate $\sigma_{33}(\mathbf{X})$ by using Eq. (23). From Eq. (20) we directly see that

$$\begin{aligned} T_{13}^+ &= -K\left[-\frac{3(\xi_1 - x_1)x_3^2}{(1-2\nu)R^5} + \frac{\xi_1 - x_1}{R^3}\right], \\ T_{23}^+ &= -K\left[-\frac{3(\xi_2 - x_2)x_3^2}{(1-2\nu)R^5} + \frac{\xi_2 - x_2}{R^3}\right], \\ T_{33}^+ &= -K\left[\frac{3x_3^3}{(1-2\nu)R^5} + \frac{x_3}{R^3}\right]. \end{aligned} \tag{25}$$

By differentiating these equations, we obtain

$$\begin{aligned} \frac{\partial T_{13}^+}{\partial x_1} &= -K\left[\frac{3x_3^2}{(1-2\nu)R^5} - \frac{1}{R^3} - \frac{15(\xi_1 - x_1)^2 x_3^2}{(1-2\nu)R^7} + \frac{3(\xi_1 - x_1)^2}{R^5}\right], \\ \frac{\partial T_{23}^+}{\partial x_2} &= -K\left[\frac{3x_3^2}{(1-2\nu)R^5} - \frac{1}{R^3} - \frac{15(\xi_2 - x_2)^2 x_3^2}{(1-2\nu)R^7} + \frac{3(\xi_2 - x_2)^2}{R^5}\right], \\ \frac{\partial T_{33}^+}{\partial x_3} &= -K\left[\frac{9x_3^2}{(1-2\nu)R^5} + \frac{1}{R^3} - \frac{15x_3^4}{(1-2\nu)R^7} - \frac{3x_3^2}{R^5}\right]. \end{aligned} \tag{26}$$

Therefore,

$$\nu\left(\frac{\partial T_{13}^+}{\partial x_1} + \frac{\partial T_{23}^+}{\partial x_2}\right) + (1-\nu)\frac{\partial T_{33}^+}{\partial x_3} = -K\left(\frac{1}{R^3} + 6\frac{x_3^2}{R^5} - 15\frac{x_3^4}{R^7}\right) \tag{27}$$

because of the second of Eqs. (21), that is

$$R^2 = (\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + x_3^2 = r^2 + x_3^2 \tag{28}$$

on the crack surface S (with $\xi_3 = 0$).

Now, by also taking into account Eqs. (19) and (21) to (24), we find that

$$\lim_{x_3 \to 0} \iint_S \frac{f(\xi_1, \xi_2)}{R^3}\left[1 + 6\left(\frac{x_3}{R}\right)^2 - 15\left(\frac{x_3}{R}\right)^4\right] \mathrm{d}S = -\frac{4\pi(1-\nu^2)}{E} p(x_1, x_2), \quad (x_1, x_2) \in S. \tag{29}$$

For $x_3 \to 0$ we find Eq. (1) as was already proved [2], where Eq. (29) was obtained on the basis of the results of Ref. [4]. From the above results it is clear that the boundary integral equation method for a plane crack under a normal pressure distribution $p(x_1, x_2)$ inside an infinite three-dimensional isotropic elastic medium leads to the singular integral equation (1) exactly as happened with the method of Bueckner [4] based on the Boussinesq–Papkovich potential.

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