Computer-aided quantifier elimination in crack problems under constraints for the stress intensity factors

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TECHNICAL NOTE

COMPUTER-AIDED QUANTIFIER ELIMINATION IN CRACK PROBLEMS UNDER CONSTRAINTS FOR THE STRESS INTENSITY FACTORS

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Abstract—It is suggested that crack problems under constraints concerning the values of the parameters involved be studied by the method of quantifier elimination in such a way that quantifier-free formulae can be derived assuring the validity of a restriction about the stress intensity factors (usually that such a factor does not exceed a critical value) for all possible values of the parameters under the applicable constraints. This permits safe conclusions about the possible fracture of a cracked specimen (under constraints on the parameters involved) for every set of values of the geometry/loading/fracture parameters. The elementary crack problem of a single straight crack in an infinite plane isotropic elastic medium is used as the vehicle for the illustration of the present approach and the related quantifier-free formula is easily derived. A simple case of a periodic array of cracks is also considered in brief. The classical algebraic tools of resultants and Sturm’s sequences (together with Sturm’s theorem) as well as the computer algebra system Maple V have been used in the present computations. More general results, with the help of Collins’ famous cylindrical algebraic decomposition method, in more complicated crack problems, can also be obtained.

INTRODUCTION

Quantifier elimination constitutes a classical tool in algebra and elementary logic permitting the derivation of formulae containing only the parameters of the related fundamental equations/inequalities which are completely equivalent to the original equations/inequalities and, moreover, free from the main variables in them and the accompanying quantifiers ∀ (universal quantifier, for all) and ∃ (existential quantifier, there exists). For example, for the real quadratic equation $a x^2 + b x + c = 0$, ...
the existence of at least one real root can be expressed, completely equivalently, by the fact that the
 corresponding discriminant \( \Delta = b^2 - 4ac \) is non-negative. We can write this fact as

\[ \exists \text{ real } x \text{ such that } \ ax^2 + bx + c = 0 \leftrightarrow \text{ discriminant}(ax^2 + bx + c) \equiv \Delta = b^2 - 4ac \geq 0. \] (1)

The formula \( \Delta \geq 0 \) in (1) is called a quantifier-free formula, since the quantifiers \( \forall \) and \( \exists \) do not appear in it.

In this note, we will illustrate the usefulness of quantifier elimination in fracture mechanics by applying this approach to a simple problem concerning stress intensity factors. This elimination will be achieved by using resultants and Sturm’s sequences (and the related theorem) [1] and the Maple V computer algebra system [2]. The present approach permits us to decide about the validity of a fact concerning stress intensity factors (which, as is well known, are of critical importance in fracture mechanics) for a set of values of the parameters involved by using only the corresponding quantifier-free formula.

We can add that Sturm’s sequences [1] were already used in elasticity in [3], but only for deciding about the truth or the falsity of a fact (with no involved parameters). On the contrary, here we will not restrict ourselves to a simple decision, but we will derive the related quantifier-free formula. Such formulae were also recently derived in [4], similarly in elasticity problems, by using the Collins method of cylindrical algebraic decomposition [5–8]. (Additional related references are also provided in [4].) Here we will use the simpler tools of resultants and Sturm’s sequences only [1] in our application below thus avoiding the somewhat more complicated cylindrical algebraic decomposition method. Moreover, we will extend quantifier elimination to fracture mechanics problems (for stress intensity factors), where, to the best of our knowledge, this method has never been used. On the other hand, in spite of the simpler tools used here (compared to the tools in [4] for a different class of problems), we will be able to simultaneously consider two parameters (instead of essentially a single parameter in the elasticity problems having been studied in [4]).

**AN APPLICATION TO STRESS INTENSITY FACTORS**

We consider the simple problem of a single straight crack \( L = [-a, a] \) inside an infinite plane isotropic elastic medium under a tensile loading \( \sigma \) at infinity of direction normal to that of the crack. Then, it is very well known that the mode I stress intensity factor \( K \) at both crack tips is given by the elementary formula [9, p. 71]

\[ K = \sigma \sqrt{\pi a}. \] (2)

We assume now that the values of \( \sigma \) and \( a \) in (1) can vary (these quantities are parameters), but that they obey a constraint, e.g.

\[ \left( \frac{\sigma}{\sigma_0} \right)^2 + \left( \frac{a}{a_0} \right)^2 \leq 1, \] (3)

where \( \sigma_0 \) and \( a_0 \) are appropriate constants, and, moreover, that the stress intensity factor \( K \) in (2) should not exceed a critical value \( K_0 \) causing the fracture of the specimen, that is

\[ K \leq K_0. \] (4)
Therefore, our present crack problem is equivalent to finding when the following statement is true

\[ (\forall \sigma, \forall a) \text{ such that } \left( \frac{\sigma}{\sigma_0} \right)^2 + \left( \frac{a}{a_0} \right)^2 \leq 1 \implies K \leq K_0. \quad (5) \]

The above statement is somewhat complicated because of the appearance of the universal quantifier \( \forall \) as well as of the related quantified variables (the parameters in our crack problem) \( \sigma \) and \( a \). We wish to find a quantifier-free formula completely equivalent to (5) in the same way that the discriminant \( \Delta \) was used in (1) as a criterion about the existence of a real root of the quadratic equation there.

To this end, we take into account the two fundamental polynomials

\[ P_1(\sigma, a) = a_0^2 \sigma^2 + \sigma_0^2 a^2 - a_0^2 \sigma_0^2 \quad \text{and} \quad P_2(\sigma, a) = \pi a \sigma^2 - K_0^2 \quad (6) \]

resulting directly from (2) to (4). By using these polynomials, we can easily find the corresponding projections on the \( a \)-axis (as is suggested in the cylindrical algebraic decomposition method [5–8]) by computing their discriminants with respect to \( \sigma \) (by using Maple’s `discrim` command [2])

\[ Q_1(a) = 4a_0^2 \sigma_0^2 (-a^2 + a_0^2) \quad \text{and} \quad Q_2(a) = 4\pi K_0^2 a \quad (7) \]

although we will not use these polynomials below. More important in our application is the resultant \( Q_3(a) \) of \( P_{1,2}(\sigma, a) \), where \( \sigma \) is eliminated from these polynomials. This polynomial, \( Q_3(a) \), was found (by using Maple’s `resultant` command [2]) to be

\[ Q_3(a) = (-\pi \sigma_0^2 a^3 + \pi a_0^2 \sigma_0^2 a - K_0^2 a_0^2)^2. \quad (8) \]

The use of all three polynomials \( Q_{1,2,3}(a) \) permits us to decompose the \( a \)-axis (according to the theory of cylindrical algebraic decomposition) and, further, decompose the \( (\sigma, a) \)-plane. But here we are just interested in the condition assuring the truth (or the falsity) of (5) and we will not construct the aforementioned decompositions. For our purpose, we simply wish to know whether the curve \( P_2(\sigma, a) = 0 \), determining the points \( (\sigma, a) \) where the stress intensity factor \( K \) becomes equal to its maximum permissible value \( K_0 \) (as is clear from (2) and (4)) has common points with the elliptical contour \( P_1(\sigma, a) = 0 \), which includes the acceptable points \( (\sigma, a) \) of the parameters \( \sigma \) and \( a \) in our application (as is clear from (3)). If there are such common points, then, obviously, (5) does not hold true, in another wording, the stress intensity factor \( K \) reaches and exceeds its maximum acceptable value \( K_0 \). On the contrary, if there are no common points of \( P_1(\sigma, a) = 0 \) and \( P_2(\sigma, a) = 0 \), then there are no points \( (\sigma, a) \) satisfying the constraint (3) such that the stress intensity factor \( K \) reaches \( K_0 \). Finally, if the curves \( P_1(\sigma, a) = 0 \) and \( P_2(\sigma, a) = 0 \) have just one common point (and a common tangent at this point), then at this point (and, evidently, only at this point) \( K \) reaches its critical value \( K_0 \) and (5) holds again true.

In order to see whether \( P_1(\sigma, a) = 0 \) and \( P_2(\sigma, a) = 0 \) have common points, we will use their resultant \( Q_3(a) \) in (8), where \( \sigma \) has been eliminated. Next, for convenience, we will use just the square root \( h_0(a) \) of \( Q_3(a) \), that is

\[ h_0(a) = \sqrt{Q_3(a)} = -\pi \sigma_0^2 a^3 + \pi a_0^2 \sigma_0^2 a - K_0^2 a_0^2. \quad (9a) \]

Based on this polynomial, we will determine the number of its distinct real zeros in the interval \((0, \infty)\) (since, obviously, the half-length \( a \) of the crack \( L \) should be always positive). This task can
be easily achieved by using Sturm's sequences and the related classical theorem [1]. The present Sturm sequence consists of \(h_0(a)\) as well as of the following polynomials

\[
h_1(a) = \frac{d h_0(a)}{da} = -3\pi\sigma_0^2 a^2 + \pi a_0^2 \sigma_0^2,
\]
(9b)

\[
h_2(a) = -\text{rem}[h_0(a), h_1(a), a] = -\frac{2}{3} \pi a_0^2 \sigma_0^2 a + K_0^2 a_0^2,
\]
(9c)

\[
h_3(a) = h_3 = -\text{rem}[h_1(a), h_2(a), a] = -\frac{4\pi^2 a_0^2 \sigma_0^4}{4\pi \sigma_0^2} - 27K_0^4,
\]
(9d)

where \(\text{rem}\) denotes the remainder of the division of the polynomials in this function (evaluated by using the related Maple's command \(\text{rem}\); similarly, the derivative in (9b) was evaluated by using Maple's \text{diff} command).

The next step is to evaluate \(h_k(a)\) \((k = 0, 1, 2, 3)\) at the ends \(a = 0\) as well as \(a = \infty\) of our open interval \((0, \infty)\) for the parameter \(c_L\). For \(a = 0\) we find easily the corresponding quantities

\[
S_0 = h_0(0) = -K_0^2 a_0^2,
\]
(10a)

\[
S_1 = h_1(0) = \pi a_0^2 \sigma_0^2,
\]
(10b)

\[
S_2 = h_2(0) = K_0^2 a_0^2,
\]
(10c)

\[
S_3 = h_3(0) = h_3 = -\frac{4\pi^2 a_0^2 \sigma_0^4}{4\pi \sigma_0^2} - 27K_0^4.
\]
(10d)

Similar is the case for \(a = \infty\), where the leading coefficients of \(h_k(a)\) (computed by using Maple's \text{lcoeff} command) should be used. These coefficients are

\[
T_0 = -\pi \sigma_0^2,
\]
(11a)

\[
T_1 = -3\pi \sigma_0^2,
\]
(11b)

\[
T_2 = -\frac{2}{3} \pi a_0^2 \sigma_0^2,
\]
(11c)

\[
T_3 = h_3 = -\frac{4\pi^2 a_0^2 \sigma_0^4}{4\pi \sigma_0^2} - 27K_0^4.
\]
(11d)

Now we observe directly from (10) and (11) that the critical quantity in our application, playing the rôle of the discriminant \(\Delta\) in (1), is the quantity

\[
D = 4\pi^2 a_0^2 \sigma_0^4 - 27K_0^4,
\]
(12)

that is the numerator of \(h_3\). In this way, three different cases arise:

(i) If \(D > 0\), then the sequence \(S_k\) in (10) has two sign variations, whereas the sequence \(T_k\) in (11) has no sign variation. Therefore, according to Sturm's theorem [1], our original polynomial \(h_0(a)\) has two distinct real zeros in \((0, \infty)\) and (5) does not hold true (there are many acceptable points \((\sigma, a)\) such that the stress intensity factor \(K\) exceeds its critical value \(K_0\)). This is the unfavourable case with (5).

(ii) If \(D < 0\), then the sequence \(S_k\) has only one sign variation and exactly the same is the case for the sequence \(T_k\). Therefore, \(h_0(a)\) has no real zeros in \((0, \infty)\). This is the favourable case in our application with (5) holding true (that is there are no acceptable points \((\sigma, a)\) such that \(K\) exceeds, or even reaches, \(K_0\)).
(iii) Finally, if $D = 0$, then the sequence $S_k$ has one sign variation again, whereas the sequence $T_k$ has no sign variation. Therefore, $h_0(a)$ has just one distinct real zero in $(0, \infty)$. This case is also favourable, since it is just the point corresponding to this zero (and only this point) where $K$ reaches its critical value $K_0$ and there is no acceptable point $(\sigma, a)$ with $K$ exceeding this value.

(Figure 1)

Because of the peculiarity of the last case (where $P_1(\sigma, a) = 0$ and $P_2(\sigma, a) = 0$ are tangent to each other at the aforementioned critical point), we display the corresponding curves (in this particular case in Fig. 1 for

$$a_0 = K_0 = 1 \quad \text{and} \quad \sigma_0 = (27/4)^{1/4}/\sqrt{\pi} \approx 0.90939$$

(13)

as is clear from (12) in the present special case, where $D = 0$. From Fig. 1 we directly observe the existence of one critical point $(\tilde{\sigma}, \tilde{a})$, which is common to the curves $P_1(\sigma, a) = 0$ and $P_2(\sigma, a) = 0$. On the other hand, the corresponding figure (not displayed here) for the first case shows that $P_1(\sigma, a) = 0$ and $P_2(\sigma, a) = 0$ cross each other (at two distinct points), whereas for the second case it shows that the aforementioned curves do not cross each other neither are they tangent at a point.

Concluding, we have found that we can eliminate the quantifier $\forall$ from (5) and that (5) is completely equivalent to

$$D = 4\pi^2 a_0^2 \sigma_0^4 - 27 \sigma_0^4 \leq 0$$

(14)

or, equivalently, to the somewhat simpler formula

$$\tilde{K}_0 \leq 1.61185 K_0$$

(15)

with $\tilde{K}_0$ defined by

$$\tilde{K}_0 = \sigma_0 \sqrt{\pi a_0}$$

(16)

a formula somewhat analogous to our fundamental formula (2).

CONCLUSIONS–DISCUSSION

From the above application we conclude that quantifier elimination methods (such as the above elementary one based on resultants and Sturm’s theorem [1] as well as the more sophisticated ones such as that based on the cylindrical algebraic decomposition method [1, 5–8]) can be successfully used in fracture mechanics problems concerning upper bounds for the stress intensity factors at crack tips. Although the above application was a rather simple one, the treatment of much more complicated crack problems does not present any theoretical difficulty although it may lead to much more complicated computations. The theoretically known exact or just approximate formulae for the stress intensity factors can be found, in most cases, in the related papers and existing general-purpose stress intensity factor handbooks.

As a slightly more complicated problem than the above one, we can mention the classical problem of a periodic array of collinear cracks (again of length $2a$) instead of a single crack above. Denoting the period of the array by $b$, we know that the mode I stress intensity factor $K$ at the crack tips (under exactly the same conditions as above) will be given by the following modification of (2) [9, p. 74]

$$K = \sigma \sqrt{\pi a} \sqrt{\tan d/d}, \quad \text{where} \quad d = \pi a/b.$$
In this crack problem, we can easily reveal that the conclusions in the previous section hold again true, but with (12) modified as

\[ D = 4\pi^2 a_0^2 \sigma_0^4 (\tan \frac{d}{d})^2 - 27K_0^4, \]  

(18)

provided that \( d = \pi a/b \) is assumed to be a constant and not a parameter. (Otherwise, we will have three parameters, \( a, b \) and \( \sigma \), in our problem, and we will have to work from the beginning in this more complicated fracture mechanics problem under the applicable constraints.) Similarly, because of (18), (16) takes the slightly more general form

\[ \tilde{K}_0 = \sigma_0 \sqrt{\pi a_0} \sqrt{\tan \frac{d}{d}} \]  

(19)

analogous to (17).

From the present results it seems obvious that quantifier-free formulae (such as (15) in both of our applications), independently of the algebraic method having been used for their derivation, permit us to directly decide whether a stress intensity factor can exceed its critical value or not in an engineering environment without having to resort to the original problem (in our applications, (5)). This decision is directly related to the possibility of fracture of the cracked medium and, therefore, the present approach seems to be not only of theoretical, but of practical importance as well.

Moreover, for those of us interested in using computer techniques in crack problems, the present results reveal also that the computer need not be used only in more or less well-known kinds of computations (such as the numerical solution of singular and hypersingular integral equations in crack problems), but it can also prove useful in less classical tasks (such as quantifier elimination above), of course, frequently equipped with the related more or less sophisticated algorithms (such as the Collins algorithm for cylindrical algebraic decomposition).

Finally, we can mention that the quantifier-free formula (14) (or the equivalent formulae (15) and (16)) have an obvious physical meaning: in order that the stress intensity factor \( K \) does not exceed its critical value \( K_0 \), it is necessary that: (i) either \( K_0 \) be sufficiently large (ii) or \( a_0 \) and \( \sigma_0 \) be sufficiently small (equivalently, because of (16), \( \tilde{K}_0 \) be sufficiently small). This was really a physically expected result confirmed here by the present approach.

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FIGURE CAPTION

Fig. 1. Relative position of the curves $P_{1,2}(\sigma, \alpha) = 0$ in the critical case (13) ($D = 0$), where the stress intensity factor $K$ reaches its critical value $K_0$ at only one acceptable point $(\bar{\sigma}, \bar{\alpha})$ satisfying (3).
(FIGURE 1)