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Application of complex path-independent integrals to locating circular holes and inclusions in classical plane elasticity

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Abstract We propose an elementary method, based on complex path-independent integrals and the classical complex potentials of Kolosov–Muskhelishvili, for the location of the position of the centre and the determination of the radius of circular holes and inclusions of a different material (either simply inserted or attached) in an infinite plane isotropic elastic medium. In practice, the method of pseudocaustics can be successfully used as the related experimental method. Generalizations of the present results follow trivially.

Keywords Plane isotropic elasticity · Complex path-independent integrals · Circular holes · Circular inclusions · Rigid/elastic inclusions · Location of holes and inclusions · Complex potentials of Kolosov–Muskhelishvili · Complex variables · Analytic functions · Meromorphic functions · Residues · Contour integrals · Cauchy's theorem · Cauchy's residue theorem · Pseudocaustics

1. Introduction

The classical problem of an infinite plane isotropic elastic medium under a unidirectional loading p at infinity and with a circular hole or a circular rigid or elastic inclusion (either simply inserted or attached) is revisited by the classical method of complex path-independent integrals. The aforementioned cases of this problem are solved in detail in the classical monograph by Muskhelishvili [1] and elsewhere. The practical usefulness of complex path-independent integrals was illustrated in a long series of papers [2–14]. Here we will not review the results of these papers, but we notice that the most fundamental ones are those proposed in Ref. [3], where the construction of an infinity of complex path-independent integrals on the basis of the Cauchy theorem in complex analysis [15] was originally proposed.

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Here we will restrict our attention to the problem of locating circular holes and inclusions in classical plane isotropic elasticity for an infinite medium by using complex path-independent integrals together with information far away from these holes and inclusions (on a closed contour C surrounding such a hole or inclusion) probably obtained by the pseudocaustics method [16]. Analogous problems for cracks in fracture mechanics were recently studied by the author [9, 10, 12] and, most probably, nowhere else.

2. The fundamental equations

For the aforementioned plane isotropic elasticity problems the classical Kolosov–Muskhelishvili complex potentials $\phi_1(z_1)$ and $\psi_1(z_1)$ are given by [1, p. 220]

$$\phi_1(z_1) = \frac{p}{4}\left(z_1 + \frac{\beta R^2}{z_1}\right), \quad \psi_1(z_1) = -\frac{p}{2}\left(z_1 + \frac{\gamma R^2}{z_1} + \frac{\delta R^4}{z_1^3}\right), \tag{1}$$

where $z_1 = x_1 + iy_1$ is the complex variable with origin of the Cartesian coordinate system coinciding with the centre of the circular hole or inclusion, p is the intensity of the uniaxial loading (along the Ox_1-axis) at infinity, R is the radius of the circular hole or inclusion and β, γ and δ are appropriate constants given in detail in Ref. [1] and depending on the problem under consideration.

In this way, in the case of a simple circular hole, we have [1, p. 218]

$$\beta = 2, \quad \gamma = 1, \quad \delta = -1. \tag{2}$$

On the other hand, in the case of a rigid inclusion joined to the surrounding plate along its edge, we have [1, p. 219]

$$\beta = -\frac{2}{\kappa}, \quad \gamma = -\frac{\kappa-1}{2}, \quad \delta = \frac{1}{\kappa}, \tag{3}$$

where κ is the Muskhelishvili constant given by [1, p. 112]

$$\kappa = 3 - 4\nu \quad \text{or} \quad \kappa = \frac{3-\nu}{1+\nu} \tag{4}$$

in the cases of plane strain and generalized plane stress, respectively. In the case of a rigid inclusion simply inserted into the hole of the matrix, we have [1, p. 219]

$$\beta = -\frac{4}{3\kappa+1}, \quad \gamma = -\frac{\kappa-1}{2}, \quad \delta = -\frac{\kappa-1}{3\kappa+1}. \tag{5}$$

Next, in the case of an elastic inclusion with Muskhelishvili's constant κ_0 joined to the matrix, we have [1, p. 221]

$$\beta = -\frac{2(\mu_0-\mu)}{\mu+\mu_0\kappa}, \quad \gamma = \frac{\mu(\kappa_0-1)-\mu_0(\kappa-1)}{2\mu_0+\mu(\kappa_0-1)}, \quad \delta = \frac{\mu_0-\mu}{\mu+\mu_0\kappa}, \tag{6}$$

where μ and μ_0 are the shear moduli (the second Lamé constants) of the matrix and the inclusion, respectively. Finally, in the case of an elastic inclusion simply inserted into the hole of the matrix, we have [1, p. 223]

$$\beta = 2\,\frac{\mu(\kappa_0+3)-2\mu_0}{\mu(\kappa_0+3)+\mu_0(3\kappa+1)}, \quad \gamma = \frac{\mu(\kappa_0-1)-\mu_0(\kappa-1)}{2\mu_0+\mu(\kappa_0-1)},$$

$$\delta = -\frac{\mu(\kappa_0+3)+\mu_0(\kappa-1)}{\mu(\kappa_0+3)+\mu_0(3\kappa+1)}. \tag{7}$$

(The remarks of Ref. [1] should also be taken into account.) Here we will not pay further attention to these values simply noting that Eqs. (1) remain valid in all cases of the present problem.

In the problem under consideration, we have to determine the centre z_0 of the circular hole or inclusion. Therefore, Eqs. (1) are not presently written in the appropriate form. Using a separate Cartesian coordinate system Oxy of our choice, where $z_0 = x_0 + iy_0$ is the sought centre, we can use the classical formulas (with $z_1 = z - z_0$) [1, p. 136]

$$\phi(z) = \phi_1(z - z_0), \quad \psi(z) = \psi_1(z - z_0) - \bar{z}_0\,\phi_1'(z - z_0). \tag{8}$$

Hence, because of Eqs. (1), we get

$$\phi(z) = \frac{p}{4}\left(z - z_0 + \frac{\beta R^2}{z - z_0}\right), \tag{9}$$

$$\psi(z) = -\frac{p}{2}\left\{z - z_0 + \frac{\gamma R^2}{z - z_0} + \frac{\delta R^4}{(z - z_0)^3} + \frac{\bar{z}_0}{2}\left[1 - \frac{\beta R^2}{(z - z_0)^2}\right]\right\}. \tag{10}$$

Two differentiations of Eqs. (9) and (10) also yield

$$\Phi(z) = \phi'(z) = \frac{p}{4}\left[1 - \frac{\beta R^2}{(z - z_0)^2}\right], \tag{11}$$

$$\Psi(z) = \psi'(z) = -\frac{p}{2}\left[1 - \frac{\gamma R^2}{(z - z_0)^2} - \frac{3\delta R^4}{(z - z_0)^4} + \bar{z}_0\,\frac{\beta R^2}{(z - z_0)^3}\right] \tag{12}$$

as well as

$$\Phi'(z) = \phi''(z) = \frac{p}{2}\,\frac{\beta R^2}{(z - z_0)^3}, \tag{13}$$

$$\Psi'(z) = \psi''(z) = -\frac{p}{2}\left[\frac{2\gamma R^2}{(z - z_0)^3} + \frac{12\delta R^4}{(z - z_0)^5} - 3\bar{z}_0\,\frac{\beta R^2}{(z - z_0)^4}\right]. \tag{14}$$

3. The approach

Here we will use the more or less classical results of the theory of analytic functions (mainly the Cauchy theorem in complex analysis [15]) for the location of poles of meromorphic functions [17]. A review of these results was recently made by the author [18]. No special study of these results is required here beyond the Cauchy theorem.

In this way, using a closed sectionally smooth contour C in the complex plane surrounding the hole or inclusion under location and assuming the necessary information for the aforementioned complex potentials available on C, on the basis of the Cauchy theorem [15], we obtain from Eq. (9)

$$\oint_C (z - z_0)\phi(z)\,\mathrm{d}z = 0. \tag{15}$$

Therefore,

$$z_0 = \frac{\oint_C z\phi(z)\,\mathrm{d}z}{\oint_C \phi(z)\,\mathrm{d}z}. \tag{16}$$

Of course, the above complex contour integrals are independent of the contour C, that is, they are complex path-independent integrals. Using the notation

$$I_j := \frac{1}{2\pi i}\oint_C z^j\phi(z)\,\mathrm{d}z, \quad j = 0, 1, \ldots, \tag{17}$$

where the integrals are evaluated in the anticlockwise direction on C, in a more general way, we have

$$z_0 = \frac{I_{j+1}}{I_j}, \quad j = 0, 1, \ldots, \tag{18}$$

since

$$\oint_C z^j (z - z_0) \phi(z)\, \mathrm{d}z = 0, \quad j = 0, 1, \ldots . \tag{19}$$

If we wish to get additional information on the hole or inclusion beyond its center z_0, we can directly compute the integral I_0 on the basis of the Cauchy residue theorem. Then we find

$$I_0 = \frac{p}{4} \beta R^2. \tag{20}$$

For holes or rigid inclusions this equation permits us to determine the radius R of the hole or the rigid inclusion. The same also happens in the case of elastic inclusions if their elastic constants are known in advance like those of the matrix. Similarly, again on the basis of the Cauchy residue theorem, we obtain from Eq. (9)

$$I_1 = \frac{p}{4} \beta R^2 z_0. \tag{21}$$

By dividing Eq. (21) by Eq. (20), we return to Eq. (16) (or, equivalently, to Eq. (18) for $j = 0$) having been obtained previously on the basis of the Cauchy theorem instead of the Cauchy residue theorem having been used for the derivation of Eqs. (20) and (21).

In a completely analogous way, if we use the complex potential $\Phi(z) = \phi'(z)$ instead of $\phi(z)$, Eq. (11) leads to

$$\oint_C z^j (z - z_0)^2 \Phi(z)\, \mathrm{d}z = 0, \quad j = 0, 1, \ldots . \tag{22}$$

Thus we have available an infinity of equations although we need only two of these for the determination of z_0. Using Eq. (22) for two different values l and m of j, because of Eq. (17), but now with $\Phi(z) = \phi'(z)$ instead of $\phi(z)$, we find that

$$I_j z_0^2 - 2I_{j+1} z_0 + I_{j+2} = 0, \quad j = l, m. \tag{23}$$

Solving this system of two linear algebraic equations with respect to z_0 and z_0^2, we obtain the required value of z_0. Frequently, we choose the values 0 and 1 for l and m, respectively, exactly as was made in Ref. [9] in a different problem. The separate solution of each one of Eqs. (23) as quadratic equations is also possible but not recommended. The common zero of these equations is the sought position z_0 of the centre of the circular hole or inclusion. Moreover, after the determination of z_0, we can use the equation

$$-\oint_C (z - z_0) \Phi(z)\, \mathrm{d}z = I_0 - I_1 = \frac{p}{4} \beta R^2 \tag{24}$$

(because of Eq. (11)) in order to determine the radius R exactly as was made in the case of the complex potential $\phi(z)$ on the basis of Eqs. (20) or (21).

Finally, if we use the derivative $\Phi'(z) = \phi''(z)$, which has the advantage of direct experimental determination in the case of generalized plane stress conditions [3, 16], Eq. (13) leads to

$$\oint_C z^j (z - z_0)^3 \Phi'(z)\, \mathrm{d}z = 0, \quad j = 0, 1, \ldots . \tag{25}$$

We have again available an infinity of equations and in this case we need only three of these equations for the determination of z_0. Using Eq. (25) for three different values l, m and n of j, because of Eq. (17), but now with $\Phi'(z) = \phi''(z)$ instead of $\phi(z)$, we find that

$$I_j z_0^3 - 3I_{j+1} z_0^2 + 3I_{j+2} z_0 - I_{j+3} = 0, \quad j = l, m, n. \tag{26}$$

Solving this system of three linear algebraic equations with respect to z_0, z_0^2 and z_0^3, we obtain the required value of z_0. Frequently, we use the values 0, 1 and 2 for l, m and n, respectively, exactly as was made in Ref. [9] in a different problem. The separate solution of each one of Eqs. (26) as cubic equations is also possible but not recommended. The common zero of these equations is the sought value of z_0. Moreover, after the determination of z_0, because of Eq. (13), we can use the equation

$$\oint_C (z-z_0)^2 \Phi'(z)\,\mathrm{d}z = I_0 z_0^2 - 2I_1 z_0 + I_2 = \frac{p}{2}\beta R^2 \tag{27}$$

in order to determine the radius R exactly as was made previously on the basis of Eqs. (20), (21) and (24).

The above results were obtained by using only $\phi(z)$ or its two first derivatives, $\Phi(z) = \phi'(z)$ and $\Phi'(z) = \phi''(z)$, and permitted us the evaluation of z_0 and βR^2 under the assumption that the loading intensity p is known in advance. The complete expressions for β were given in detail in the previous section: for example, $\beta = 2$ in the case of a circular hole. Furthermore, it is possible to use the second complex potential $\psi(z)$, given by Eq. (10) in our set of problems, or its two first derivatives $\Psi(z) = \psi'(z)$ and $\Psi'(z) = \psi''(z)$ given by Eqs. (12) and (14), respectively, in the same problems. Here we will consider only the case of $\psi(z)$. The consideration of the cases of $\Psi(z) = \psi'(z)$ and $\Psi'(z) = \psi''(z)$ is completely analogous exactly as previously with $\phi(z)$, $\Phi(z) = \phi'(z)$ and $\Phi'(z) = \phi''(z)$.

In this case, by taking into account that z_0 is now a known quantity and defining the complex path-independent integrals (again in the anticlockwise direction)

$$J_j := \frac{1}{2\pi i}\oint_C (z-z_0)^j \psi(z)\,\mathrm{d}z, \quad j = 0, \pm 1, \pm 2, \dots, \tag{28}$$

by using Eq. (10), from the Cauchy residue theorem [15] we obtain

$$J_{-2} = -\frac{p}{2}, \quad J_{-1} = -\frac{p\bar{z}_0}{4}, \quad J_0 = -\frac{p}{2}\gamma R^2, \quad J_1 = \frac{p\bar{z}_0}{4}\beta R^2, \quad J_2 = -\frac{p}{2}\delta R^4,$$
$$J_j = 0, \quad j = \pm 3, \pm 4, \dots\,. \tag{29}$$

In Eq. (28), we have used both positive and negative values of j. The same is possible for the complex potential $\phi(z)$ and its derivatives as well. We observe that in our problem we can obtain valuable information for the quantities of interest for $j = 0, \pm 1$ and ± 2. We will not enter into further more or less trivial details.

4. Discussion

The method of complex path-independent integrals used above for the set of problems under consideration has the important advantage that it gathers information on a whole closed contour C surrounding the point or region of interest. Therefore, this method is interesting not only in experimental approaches but also in numerical approaches such as the finite-element method, which become rather inaccurate near regions of high stress concentrations. On the contrary, the determination of quantities of interest (such as z_0 above) by using experimental or numerical results from concrete points of the isotropic elastic medium and not from a whole closed contour C (as has been the case here with the closed contour C) may lead to significant errors.

On the other hand, we have applied the method of complex path-independent integrals to the simple problems of the theory of plane isotropic elasticity concerning the location of circular holes or inclusions for an infinite medium and, approximately, for a finite medium of dimensions much larger than the radius R of the hole or the inclusion. It is clearly possible that the present method can be generalized without difficulty to more complicated problems such as the problems of holes or inclusions of more complicated shapes, e.g. elliptical holes or inclusions. Of course, in such cases, the necessary algebraic manipulations and the real computations for the application of the present method increase sometimes considerably.

Finally, the problem of determination of the shape of the hole or inclusion (assumed arbitrary and not known in advance) seems to be a very difficult problem having not been tackled so far either by the present method of complex path-independent integrals or by any other method (to this author's best knowledge), yet worthy of particular attention by researchers in elasticity. The possibility of solution of this problem by the method of complex path-independent integrals seems not only difficult but in principle doubtful as well. Of course, in the latter case, the probable inability of the method to be used for the solution of the aforementioned general and important problem has to be proved mathematically.

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