On the efficient computation of the stress components near a closed boundary in plane elasticity by using classical complex boundary integral equations

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ON THE EFFICIENT COMPUTATION OF THE STRESS COMPONENTS NEAR A CLOSED BOUNDARY IN PLANE ELASTICITY BY USING CLASSICAL COMPLEX BOUNDARY INTEGRAL EQUATIONS

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ABSTRACT

Complex boundary integral equations (Fredholm-type regular or Cauchy-type singular or even Hadamard-Mangler-type hypersingular) have been used for the numerical solution of general plane isotropic elasticity problems. The related Muskhelishvili and, particularly, Lauricella–Sherman equations are famous in the literature, but several more extensions of the Lauricella–Sherman equations have also been proposed. In this paper it is just mentioned that the stress and displacement components can be very accurately computed near either external or internal simple closed boundaries (for anyone of the above equations: regular or singular or hypersingular, but with a prerequisite their actual numerical solution) through the appropriate use of the even more classical elementary Cauchy theorem in complex analysis. This approach has been already used for the accurate numerical computation of analytic functions and their derivatives by Ioakimidis, Papadakis and Perdios [BIT, 31, 276–285 (1991)], without applications to elasticity problems, but here the much more complicated case of the elastic complex potentials is studied even when just an appropriate non-analytic complex density function (such as an edge dislocation/loading distribution density) is numerically available on the boundary. The present results are also directly applicable to inclusion problems, anisotropic elasticity, antiplane elasticity and classical two-dimensional fluid dynamics, but, unfortunately, not to crack problems in fracture mechanics. Brief numerical results (for the complex potentials), showing the dramatic increase of the computational accuracy, are also displayed and few generalizations are proposed. © 1998 assigned to John Wiley & Sons, Ltd.

KEY WORDS: boundary integral equations; Cauchy-type integrals; Cauchy theorem; elasticity; numerical integration; singular/hypersingular integral equations

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1. INTRODUCTION—THE CAUCHY THEOREM/FORMULA

The Cauchy theorem as well as the (related) Cauchy integral formula are extremely well known and useful in classical complex analysis. Moreover, they were already used thousands of times during the efficient solution of applied mechanics and engineering problems. More explicitly, for an analytic function \( f(z) (z = x + iy) \), defined at every point \( z \) of a finite open region \( D \) of the complex plane as well as on its boundary \( C = \partial D \) (assumed a simple and closed contour), the Cauchy theorem states that

\[
\oint_C f(t) \, dt = 0.
\]

(We denote the points \((x, y) = x + iy\) of \( D \) by \( z \) and the points of \( C \) either by \( t \) or by \( \tau \).)

Moreover, the Cauchy integral formula is somewhat more efficient from the practical point of view, since it permits the computation of the analytic function \( f(z) \) in the whole (open) region \( D \) surrounded by \( C \) through the following equation (with the integration in the positive direction):

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} \, dt, \quad z \in D.
\]

The meaning of this formula is simply that once the boundary values \( f(t) (t \in C) \) of an analytic function \( f(z) \) (in \( \bar{D} = D \cup C \)) are known, then we can directly compute \( f(z) \) at any point \( z \in D \) essentially by just computing the contour integral in (2). Moreover, successive differentiations of (2) yield

\[
f^{(m)}(z) = \frac{m!}{2\pi i} \oint_C \frac{f(t)}{(t-z)^{m+1}} \, dt, \quad m = 1, 2, \ldots, \quad z \in D.
\]

No doubt, these are extremely classical results.

In a very recent (summer 1998) paper, Martin, Rizzo and Cruse considered in some detail the use of the Cauchy integral formula (2) and its first derivative (for \( m = 1 \) in (3)) in smoothness–relaxation strategies for singular and hypersingular integral equations. More explicitly, they used the formula (in the present notation)

\[
\oint_C \frac{f(t)}{t-z} \, dt = 0, \quad z \in \partial D,
\]

as well as the first-derivative related analogous formula

\[
\oint_C \frac{f(t) - f'_{\tau}}{(t - \tau)^2} \, d\tau = 0, \quad \tau \in C,
\]

on the boundary \( C \) of the region \( D \) for the regularization of the Cauchy-type singularity in the Cauchy formulae (2) and (3) (for \( m = 1 \)), respectively. This is very useful for the regularization of singular and hypersingular CBIEs (complex boundary integral equations). In fact, from another point of view, both formulae (4) and (5) can also be viewed as simple applications of the Cauchy theorem (in lieu of the Cauchy integral formula) for the integrands in the fractions of these formulae, which are, obviously, analytic functions provided that \( f(z) \) was already assumed to be such a function in \( \bar{D} = D \cup C \).

In this paper we will consider the use of (4) and (5) further, but from a completely different point of view. In fact, we will try to show the usefulness of these formulae (but for \( z \in D \) instead of \( t \in C \)) during the actual efficient numerical computation of the stress and displacement components in plane elasticity with \( D \) being a plane isotropic elastic medium (or in analogous problems related to CBIEs).
This will be done after the actual numerical solution of the related CBIE, which will not be a topic of study in this paper. Beyond the use of (4) and (5) for boundary values $t$ of $z$, which may be (although rarely) necessary as will be seen below (even after the definitive numerical solution of the CBIE), we will also (and mainly) show the usefulness of the same formulae for $z \in D$ (and not only on $C$), i.e. inside the elastic medium $D$ itself.

On the other hand, it should also be mentioned that the effective computation of quantities of mechanical interest (mainly stress and displacement components) has been the topic of intensive research inside the popular BEM (boundary element method) long ago (see, e.g., References 3–9), but with respect to real BIEs (boundary integral equations) and, moreover, without the use of the Cauchy theorem. The purpose of these references has been mainly the deletion of the so-called ‘boundary-layer effect’ and, in fact, this purpose is completely analogous to that in the present paper (but for the BEM there instead of CBIEs here). The sole reference actually related to CBIEs and with an aim similar to that in the present paper in plane elasticity that this referee is aware of is the paper by Theotokoglou, but the Cauchy theorem (1) has not been used even in this paper (although Cauchy-type integrals have been extensively used). This author’s opinion is simply that the use of the Cauchy theorem (directly or in any equivalent way) is the sole approach naturally leading to the complete elimination of the singularity in Cauchy-type integrals for the complex potentials at a point $z \in D$, but near its boundary $C$. This is the approach exclusively adopted in this paper and the author believes that there is no more efficient approach for the elimination of the boundary-layer effect in CBIEs.

The origin of the method employed in the present paper can be found in a paper by Ioakimidis, Papadakis and Perdios, but in that paper no implementation of the method to elasticity (and related) problems was made. Moreover, neither the formulae (4) and (5) were used for boundary points $t$ of the elastic region $D$ contrary to the aforementioned paper of Martin, Rizzo and Cruse, where boundary points were used. Here we will use again these formulae, although for interior points $z$ of the elastic medium $D$, in an attempt to describe the possibility of a very accurate numerical computation of the stress and displacement components inside the computational environment offered by classical CBIEs in plane elasticity since more than half a century. We believe that this regularization approach (of the Cauchy-type singularities) has not received the attention that it would deserve in computational mechanics and engineering (even Martin, Rizzo and Cruse did not exploit it as much as it would be possible), neither its practical implementation to the numerical evaluation of the stress and displacement components has been illustrated. This will be done in the present paper, where, moreover, we intend to generalize (in Section 6) the results of Reference 11 (exclusively referring to boundary values $f(t)$, on $C$, of analytic functions $f(z)$ in $\bar{D} = D \cup C$) to non-analytic density functions $g(t)$ (on $C$) in Cauchy-type integrals. Finally, the important case of the generalization of the present approach to the modern BEM (boundary element method), even if possible in principle, is simply being left to the experts (with just a related hint in Section 8 of this paper), the education of this author being extremely classical in this respect.

We will show below that any regular, singular or hypersingular CBIE in plane elasticity that we have in mind can be combined with the subsequent use of (4) and (5) (but with $z \in D$ instead of $t \in C$) when actual numerical computations of the stress and displacement fields near the closed boundary $C$ of the elastic region $D$ is required. The present results will be based on classical, interpolatory numerical integration rules over closed contours $C$, their potential influence on alternative approaches (such as the BEM) being doubtful and, surely, not implemented here as was already mentioned. Moreover, the case of open boundaries $L$ (mainly cracks or rigid/flexible line inclusions) is also not studied and the present approach seems to completely fail in this particular case.

With respect to CBIEs (complex boundary integral equations), we can mention that these are extremely popular in plane isotropic elasticity and related fields (mainly in plane anisotropic elasticity and
antiplane elasticity, but also in classical fluid dynamics and potential theory) long ago. The interested reader can consult the classical monographs by Muskhelishvili, Babuška, Rektorys and Vyčichlo, Sokolnikoff, Kalandiya, Savruk and Parton and Perlin as far as CBIEs are concerned. Related Cauchy-type integrals can also be found in these monographs as well as in the monograph by Milne-Thomson.

For the theory of fluid dynamics reference can be made to the monograph by Milne-Thomson and for anisotropic elasticity to that by Lekhnitskii. CBIEs in fluid dynamics (for airfoils) can also be found in the monograph by Mikhlin and that by Muskhelishvili, the latter being the standard reference for the theory of Cauchy-type singular CBIEs and Cauchy-type integrals in general.

From a different point of view and restricting our attention to static plane linear isotropic elasticity, we can mention that the first two classical related methods are those due to Muskhelishvili (derived in 1933–34 and leading to a CBIE, but see also Reference 23 by the same author) and to Sherman: the popular Lauricella–Sherman CBIE, based on real integral equations by Lauricella (derived in 1909), having been considerably modified, extended and brought to the form of CBIEs by Sherman (derived about 1940). The CBIE of Sherman is much more popular than that of Muskhelishvili simply for computational reasons. Both of these methods will be considered in some detail below, but just as far as the computation of the stress and displacement components near a simple closed boundary is concerned.

There is also an extremely large number of more recent results concerning CBIEs in plane elasticity. Among them we can make reference, e.g., to the original paper by Lin'kov, having been extended and published in several papers, as well as to the (generally) much more recent further results by Lin'kov and Mogilevskaya and Chen and his collaborators. A very large number of additional related research papers is also available in the literature (see, e.g., References 54 and 55). What should also be mentioned is that during the last ten years there is a tendency (due just to physical and computational reasons) to move from Cauchy-type singular CBIEs to Hadamard–Mangler-type (with a second-order pole) hypersingular (finite-part) CBIEs in plane elasticity. Below (in Section 7) we will pay an equal attention to both singular and hypersingular CBIEs (no discrimination!).

Of course, beyond the 'pure' CBIEs in elasticity, fluid dynamics and engineering in general, CBIEs have also been extensively used in the CVBEM (complex variable boundary element method) by Hromadka II, his collaborators and additional researchers (just to make reference to a book and few of the related papers). Yet here, contrary to the BEM (boundary element method), we will assume the use of ordinary quadrature rules (for simple closed contours in the complex plane). The interested reader can consult the classical monograph by Davis and Rabinowitz, (including a paragraph on numerical integration on closed contours, mainly by the use of the classical trapezoidal quadrature rule to be used here as well), the computer/software-oriented book by Piessens, de Doncker-Kapenga, Überhuber and Kahaner, (including a package for real Cauchy-type integrals) and additional books and papers. The existence of poles near the integration interval (as is here the case, but for a closed contour) has also been studied in detail even for closed contours, whereas an interesting, very recent approach to numerical integration in the presence of complex singularity poles is described by Dumont and Noronha. Moreover, closed contour integration is frequently based on quadrature rules for periodic functions and there is also a large number of related (but not necessarily based just on the trapezoidal quadrature rule) interesting references.

2. THE FUNDAMENTAL COMPLEX-VARIABLE EQUATIONS

As was already mentioned, we will restrict our attention to classical static plane linear isotropic
elasticity. We will use the extremely well-known Kolosov–Muskhelishvili complex potentials \( \phi(z) \) and \( \psi(z) \) for the determination of the displacement components \( (u \text{ and } v) \) and the stress components \( (\sigma_x, \sigma_y \text{ and } \sigma_{xy}) \) inside the elastic medium \( D \) bounded by the contour \( C \equiv \partial D \). The related equations have the form\(^{12}\)

\[
2\mu(u + iv) = \kappa \phi(z) - z\phi'(z) - \psi(z),
\]

\[
\sigma_x + \sigma_y = 2[\phi'(z) + \phi''(z)] = 4\Re\phi'(z),
\]

\[
\sigma_y - \sigma_x + 2i\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)],
\]

where \( \mu \) denotes the shear modulus and \( \kappa \) the Muskhelishvili constant of the elastic material. Quite frequently, especially when we intend to study only the stress components, the first derivatives of the Kolosov–Muskhelishvili complex potentials\(^{12}\)

\[
\Phi(z) = \phi'(z), \quad \Psi(z) = \psi'(z)
\]

are used in the last two of the above fundamental formulae.

Our plan is simply to have available very accurate values for the complex potentials \( \phi(z) \) and \( \psi(z) \) and their derivatives \( \phi'(z) \), \( \phi''(z) \) and \( \psi'(z) \) so that we can efficiently use (6) to (8) for the computation of equally very accurate values of the stress and displacement components in \( D \). Unfortunately, for points \( z \) of the medium \( D \) near the boundary \( C \), this is not easily possible when we have to use the Cauchy integral formula (2) and its extensions (3) for the computation of these analytic functions in \( D \) on the basis of their boundary values on \( C \) (under the obvious assumption that a CBIE has been already constructed and actually solved on \( C \)). Therefore, the Cauchy theorem (1) has to be used (instead) for interior points \( z \) of \( D \) (but near \( C \)) as was already made by Ioakimidis, Papadakis and Perdios.\(^{12}\)

Unfortunately, the situation is somewhat more complicated here since, in the best case, in CBIEs the boundary values of only one complex potential (generally \( \phi(z) \)) are available and, what is even worse, this happens only in the method of Muskhelishvili (to be studied in Section 3 just below). Therefore, the boundary values \( \psi(t) \) \( (t \in C) \) of \( \psi(z) \) have also to be computed and this is not completely trivial. Moreover, what is even more important in the Lauricella–Sherman Fredholm CBIE and, in general, in all (classical) singular and (modern) hypersingular competitive CBIEs as well, the Kolosov–Muskhelishvili complex potentials \( \phi(z) \) and \( \psi(z) \) and/or their derivatives are expressed in terms of density functions (such as edge dislocation distributions) on the boundary \( C \) of \( D \), which have nothing to do with analytic functions (and, therefore, with the Cauchy theorem and the related integral formula). In this author’s personal opinion, these may have been the main reasons that the Cauchy theorem seems not to have been used so far for the accurate computation of the stress and displacement components near the boundary \( C \) of \( D \).

In the next section (Section 3), we will clarify the present approach in the case of the Muskhelishvili Fredholm CBIE, in Section 4 we will display the results of some elementary numerical experiments (illustrating the dramatic increase in accuracy at points \( z \) of \( D \) near the closed boundary \( C \)), in Section 5 we will refer to the classical Lauricella–Sherman CBIE, in Section 6 we will study the case of arbitrary, non-analytic density functions \( g(t) \) on \( C \) (displaying also further numerical results), in Section 7 we will refer to singular/hypersingular CBIEs in general and, finally, in Section 8 we will report our conclusions as well as direct or simply possible generalizations of the approach.

Before proceeding further, we can also add that a first attempt towards the accurate computation of complex potentials \( f(z) \) (in plane elasticity) in the case of use of a density function \( g(t) \) (on \( C \)) and of a Cauchy-type integral of the form
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\[ f(z) = \frac{1}{2\pi i} \oint_C \frac{g(t)}{t-z} \, dt, \quad z \in D, \]  

(10)

(not just of the Cauchy integral formula (2), where \( g(t) \equiv f(t) \)) was made (as was already mentioned) by Theotokoglou,\(^{10}\) who studied in detail the error term due to the first-order singularity in the above formula (10) when using the trapezoidal quadrature rule for a circular domain \( D \). Unfortunately, beyond the restriction to a circular domain \( D \) in these results,\(^{10}\) the type of \( g(t) \) in (10) (analytic or non-analytic) has not been clarified by Theotokoglou\(^{10}\) in a satisfactory way, but both in the construction of the interpolatory quadrature rule there (p. 2116 of Reference 10) and in the discussion section of the same reference (p. 2128), where the phrase 'provided the analytic value for the function \( f(\sigma) \) at \( z \) is known' is included (with \( f(\sigma) \) coinciding with \( g(t) \) in (10) in the present notation), give the reader the clear impression that these results by Theotokoglou,\(^{10}\) although both original and very interesting, yet concern only analytic density functions \( g(t) \) in spite of the use of (10) (in (1) of Reference 10, further in (17) there for the complex potentials \( \phi(z) \) and \( \psi(z) \) themselves).

Here we intend to generalize the results of Theotokoglou\(^{10}\) by considering any interpolatory quadrature rule (in spite of the fact that the trapezoidal quadrature rule will be used again in the numerical experiments of Sections 4 and 6) and any sectionally smooth geometry of the closed boundary \( C \) of the elastic medium \( D \) (not only a circle) and, what is also important, by using the Cauchy theorem instead of the Cauchy integral formula. Moreover, a clear distinction (not made by Theotokoglou\(^{10}\)) between the cases when \( g(t) \) is (i) the boundary value \( f(t) \) of \( f(z) \) in (10) itself or (ii) just an independent density function (having nothing to do with analytic functions contrary to \( f(z) \)) will be made below (Sections 3–4 and 5–7, respectively) and, in general, we will attempt to attack the problem of the \emph{complete disappearance} of the complex pole in the Cauchy-type integrals, which is present essentially in any available CBIE (either Fredholm-type regular or Cauchy-type singular or, finally/recently, Hadamard–Mangler-type hypersingular, either by using the Kolosov–Muskhelishvili complex potentials \( \phi(z) \) and \( \psi(z) \) or their derivatives \( \Phi(z) \) and \( \Psi(z) \)) in a definitive and (in our opinion) satisfactory as well way, yet leaving the (non-trivial) task of generalizations to the boundary element CBIEs (in the CVBEM)\(^{56–63}\) (or, further, to the ordinary BEM) to the experts in boundary elements.

3. THE MUSKHELISHVILI COMPLEX BOUNDARY INTEGRAL EQUATION

This is the simplest approach in plane elasticity and has to do just with a Fredholm essentially regular CBIE on the boundary \( C \) of \( D \) (reducible also to a system of two real Fredholm integral equations). More explicitly, for both the first (traction-related) and the second (displacement-related) boundary value problems, the related complex boundary condition on \( C \) is\(^{12}\)

\[ \bar{k}\overline{\Phi(t)} + \bar{t}\overline{\phi'(t)} + \overline{\psi(t)} = \overline{p(t)}, \quad t \in C, \]  

(11)

where \( p(t) \) is a completely known non-analytic function on the boundary \( C \), based either on the traction (after an indefinite integration) or on the displacement components\(^{12}\) and \( k = 1 \) (\( k = -\kappa \)) for the first (second) boundary value problems, respectively.\(^{12}\) The related Fredholm CBIE has the following form\(^{12}\) (in the present notation and after having taken the complex conjugate of this CBIE):

\[ k\phi(t) - \frac{k}{2\pi i} \oint_C \phi(\tau) \, d\log \frac{\tau - t}{\tau - i} - \frac{1}{2\pi i} \oint_C \overline{\phi(\tau)} \, d\frac{\tau - t}{\tau - i} = -A(t), \quad t \in C, \]  

(12)

where, in the right-hand side function, \( A(t) \) is determined from\(^{12}\)
Here we will simply assume that the Muskhelishvili Fredholm CBIE (12) has been already solved by using some appropriate numerical method and that the boundary values \( \phi(t) \) of \( \phi(z) \) \( (t \in C) \) have been approximately determined. We will denote these approximate values by \( \phi(t) \). We are now interested in the determination of the stress and displacement components at an interior point \( z \in D \) by using these values and, evidently, the Kolosov–Muskhelishvili complex potentials \( \phi(z) \) and \( \psi(z) \). Surely, this should be done on the basis of the fundamental formulae (6) to (8) in the previous section. On the basis of these formulae, it is clear that we require sufficiently accurate values of \( \phi(z) \), \( \phi'(z) \), \( \phi''(z) \), \( \psi(z) \) and \( \psi'(z) \) (at the point \( z \) of \( D \), probably close to the boundary \( C \), under consideration) so that all of the stress and displacement components can be (approximately, but also sufficiently accurately) computed.

In principle, for this task we have simply to use the Cauchy integral formula (2) and its derivatives (3) as far as the complex potential \( \phi(z) \) is concerned, i.e.

\[
\tilde{\phi}(z) = \frac{1}{2\pi i} \oint_C \frac{\tilde{\phi}(t)}{t-z} \, dt, \quad \tilde{\phi}'(z) = \frac{1}{2\pi i} \oint_C \frac{\tilde{\phi}(t)}{(t-z)^2} \, dt, \quad \tilde{\phi}''(z) = \frac{1}{\pi i} \oint_C \frac{\tilde{\phi}(t)}{(t-z)^3} \, dt, \quad z \in D, \tag{14}
\]

(the tilde again simply being used to denote the approximate character of the numerical solution of the Muskhelishvili Fredholm CBIE (12)).

For the numerical computation of \( \psi(z) \) and \( \psi'(z) \) the situation is slightly more difficult, but we can use again the Cauchy integral formula (2) and the first of (3), i.e.

\[
\tilde{\psi}(z) = \frac{1}{2\pi i} \oint_C \frac{\tilde{\psi}(t)}{t-z} \, dt, \quad \tilde{\psi}'(z) = \frac{1}{2\pi i} \oint_C \frac{\tilde{\psi}(t)}{(t-z)^2} \, dt, \quad z \in D, \tag{15}
\]

(again with approximate values). The slight difficulty with \( \psi(z) \) and \( \psi'(z) \) lies in the fact that it is \( \phi(t) \) that has been the unknown function in the present CBIE (12) (not \( \psi(t) \)). Yet, we can easily determine \( \tilde{\psi}(t) \) as well on the basis of the already determined \( \tilde{\phi}(t) \) by using the boundary condition (11). Then we directly get

\[
\tilde{\psi}(t) = \overline{p(t)} - k\overline{\phi(t)} - i\overline{\phi'(t)}, \quad t \in C, \tag{16}
\]

and, therefore, the Cauchy integral formulae (15) can, in principle, be used (as is also really suggested by Muskhelishvili\textsuperscript{12} for \( \psi(z) \)).

Evidently, the function \( p(t) \) on the boundary \( C \) is assumed known/determined in advance, the unknown function \( \tilde{\phi}(t) \) has been determined from the numerical solution of the CBIE (12) and, in this way, it is only \( \tilde{\phi}'(t) \) \( (t \in C) \) that is still unknown and enters into (16) and, therefore, it causes a slight difficulty in the whole approach, since this function has not been directly determined during the numerical solution of (12). For the numerical determination of \( \tilde{\phi}'(t) \) there are four possibilities: (i) The direct numerical differentiation of \( \tilde{\phi}(t) \). Yet this possibility may lead to a significant loss of accuracy, since numerical differentiation is a low-accuracy method in numerical analysis. (ii) The numerical differentiation of the CBIE (12) itself (evidently with respect to \( t \)) and its use (after the approximate computation of \( \tilde{\phi}(t) \)) for the computation of \( \tilde{\phi}'(t) \) as well, now appearing as its first term. This seems to be a very good approach. (iii) The use of the Cauchy theorem in (4) and (5) at first for the computation of \( \tilde{\phi}(t) \) at additional points \( t_j^* \) of \( C \) (through the use of (4)) and, next, for
computation of $\tilde{\Phi}'(t)$ at the original points $t_j$ of computation of $\Phi(t)$ (through the use of (5)). (This is a useful consequence of the aforementioned results by Martin, Rizzo and Cruse)\(^2\) (iv) Finally, perhaps the best possibility of all is to use the following formula, already suggested by Muskhelishvili\(^1\), but, obviously, with exact, not approximate values in that reference:

$$
\Psi(z) = \frac{1}{2\pi i} \oint_C \frac{p(t)}{t-z} \, dt - \frac{k}{2\pi i} \oint_C \frac{\Phi(t)}{t-z} \, dt - \frac{1}{2\pi i} \oint_C \frac{i\Phi'(t)}{t-z} \, dt, \quad z \in D,
$$

simply resulting from the substitution of $\Psi(t)$ from (16) into the first of (15). Then an integration by parts in the third contour integral of the last equation, (17), directly gives a formula able to determine $\Psi(z)$ on the basis only of the values of $p(t)$ and $\Phi(t)$. Then (17) takes the following much more appropriate form:

$$
\Psi(z) = \frac{1}{2\pi i} \oint_C \frac{p(t)}{t-z} \, dt - \frac{k}{2\pi i} \oint_C \frac{\Phi(t)}{t-z} \, dt + \frac{1}{2\pi i} \oint_C \frac{\Phi(t)}{t-z} \, dt - \frac{1}{2\pi i} \oint_C \frac{i\Phi'(t)}{(t-z)^2} \, dt, \quad z \in D,
$$

where $\Phi'(t)$ does not appear any more. Finally, as far as $\Psi'(z)$ is concerned, simply we can differentiate the above Cauchy integral formula (18) with respect to $z$ on the understanding that a third-order pole will appear in the last Cauchy-type integral exactly as has been already the case in the third of (14) (that for $\Phi''(z)$).

The conclusion of this section is simply that the complete and accurate (near the boundary $C$ of the elastic region $D$) determination of the stress and displacement components by using the Muskhelishvili Fredholm CBIE is possible, provided that we are able to accurately compute the values of Cauchy-type integrals and their first derivatives (exactly as in (2) and (3) with $m = 1, 2$). This task can be achieved on the basis of the Cauchy theorem (1) according to the approach by Ioakimidis, Papadakis and Perdios\(^1\) essentially consisting in the regularization of the Cauchy integral formula (2) and its derivatives (3), analogously to what was very recently made by Martin, Rizzo and Cruse\(^2\) but for $z \in D$ (in Reference 11) and not on its boundary $C$ (as is the case in (4) and (5)). In the next section we will display few related numerical results, based on the approach by Ioakimidis, Papadakis and Perdios\(^1\) and concerning the complex potential $\Phi(z)$ and its first two derivatives $\Phi'(z)$ and $\Phi''(z)$ for an elliptical region $D$. Several theoretical results, mainly for a circular region $D$ and when the trapezoidal quadrature rule is used, were also derived in the same paper\(^1\).

### 4. NUMERICAL EXPERIMENTS FOR THE COMPLEX POTENTIALS

We consider the problem of an elliptical region $D$ in the $Oxy$-plane with semi-axes $a = 1$ and $b = 1/2$ and centre the origin $O = (0, 0)$ of the Cartesian coordinates. The equation of its boundary $C$ is $x/a)^2 + (y/b)^2 = 1$ and, obviously, the points $t$ of this boundary are determined from

$$
t = x + iy = a \cos \theta + ib \sin \theta, \quad \theta \in [0, 2\pi),
$$

whence

$$
dt = dx + idy = (-a \sin \theta + ib \cos \theta) \, d\theta, \quad \theta \in [0, 2\pi).
$$

In our present application we aim just at showing the dramatic increase of the accuracy when the Cauchy theorem is used instead of the Cauchy integral formula for points $z$ of the elastic medium $D$ near the boundary $C$. This medium, $D$, is simply assumed to be loaded by a uniform tensile loading
$p = 2$ on its boundary $C$. Then the expressions of the first Kolosov–Muskheilishvili complex potential $\phi(z)$, its first two derivatives and the complex potential $\psi(z)$ will have the following obvious and extremely simple (but sufficient for our present purpose) forms:

$$\phi(z) = \frac{pz}{2} = z, \quad \phi'(z) = \frac{p}{2} = 1, \quad \phi''(z) = 0, \quad \psi(z) = 0. \tag{21}$$

We assume that we have already numerically determined the boundary values $\phi(t) = t$ of $\phi(z)$ (e.g. from the closed-form/numerical solution of the Muskheilishvili CBIE (12) in the previous section) and we are now interested just in the numerical computation of the complex potential $\phi(z)$ and its first two derivatives at a point $z$ inside $D$, but near the boundary $C$. It is understood that in the present elementary application, it will be only the error due to numerical integration for $\phi(z)$ and its first two derivatives (if any) that will be present in these numerical values, since the exact expression of $\phi(t) (t \in C)$ was already assumed available. This will permit us to concentrate on the study of the influence of the substitution of the Cauchy integral formula by the Cauchy theorem during the numerical computations with any errors in the numerical solution of the CBIE simply neglected. In fact, in this paper we will pay no attention at all to the improvement of the accuracy of the methods of numerical solution of CBIEs.

In general, an interpolatory quadrature rule on a closed contour $C$ with $n$ nodes $t_{jn}$ and corresponding weights $A_{jn}$ has the following general classical form (here with an additional factor $1/(2\pi i)$):}

$$\frac{1}{2\pi i} \oint_C h(t) \, dt = \sum_{j=1}^{n} A_{jn} h(t_{jn}) + E_n(h), \tag{22}$$

$E_n(h)$ denoting the error term. (Evidently, the integrand $h(t)$ need not be an analytic function in $\overline{D} = D \cup C$.)

When using the trapezoidal quadrature rule for an elliptical contour $C$, we simply select the nodal values $\theta_{jn}$ of the parameter $\theta$ in (19) and (20) equispaced on $[0, 2\pi]$, i.e.

$$\theta_{jn} = 2(j - 1)\pi/n, \quad j = 1, 2, \ldots, n, \tag{23}$$

whence, because of (19) for the elliptical contour $C$,

$$t_{jn} = a \cos \theta_{jn} + ib \sin \theta_{jn}. \quad j = 1, 2, \ldots, n. \tag{24}$$

On the other hand, the ordinary and equal weights $2\pi/n$ of the trapezoidal quadrature rule (with $n$ nodes and for periodic functions) on the real interval $[0, 2\pi]$ take, because of (20) and (22), the following form (again for the present elliptical contour $C$):

$$A_{jn} = \frac{1}{n!} (-a \sin \theta_{jn} + ib \cos \theta_{jn}), \quad j = 1, 2, \ldots, n. \tag{25}$$

We are now ready to actually apply the trapezoidal quadrature rule (22). For the Cauchy integral formula (2) and its generalizations (3) as well, this is trivial, i.e.

$$f^{(m)}(z) = m! \sum_{j=1}^{n} \frac{A_{jn} f(t_{jn})}{(t_{jn} - z)^{m+1}} + E_n^*(f^{(m)}), \quad m = 0, 1, 2, \ldots, \quad z \in D, \tag{26}$$
Table I. Numerical results for the complex potential $\phi(z)$ and its first two derivatives $\phi'(z)$ and $\phi''(z)$ for an elliptical region $D$ (with $a = 1$ and $b = 1/2$) (i) under a uniform tensile loading with $p = 2$ whence $\phi_1(t) = pt/2 = t$ (subscript 1, $\phi_1(z)$) and (ii) under a variable loading such that $\phi_2(t) = \cosh t$ (subscript 2, $\phi_2(z)$), computed at the point $z = 0.90$ on the basis of the Cauchy integral formula (values with a tilde) and the Cauchy theorem (values with a hat) and, moreover, on the basis of the exact values of $\phi(t)$ along the boundary $C$ of $D$ by using the trapezoidal quadrature rule for closed contours with $n$ nodes.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\phi_1(0.90)$</th>
<th>$\phi'_1(0.90)$</th>
<th>$\phi_2(0.90)$</th>
<th>$\phi''_1(0.90)$</th>
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<tbody>
<tr>
<td>4</td>
<td>1.39647 46772 592</td>
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<td>2.23734 62756 496</td>
<td>1.44192 49277 307</td>
</tr>
<tr>
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<td>0.9</td>
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<td>1.43308 86363 418</td>
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<tr>
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<td>0.91211 21694 476</td>
<td>0.9</td>
<td>1.45237 28133 750</td>
<td>1.43308 63854 488</td>
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<tr>
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<tr>
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<tr>
<td>32</td>
<td>1.02091 14152 188</td>
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<tr>
<td>64</td>
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<table>
<thead>
<tr>
<th>$n$</th>
<th>$\phi''''_1(0.90)$</th>
<th>$\phi''''_1(0.90)$</th>
<th>$\phi''''_2(0.90)$</th>
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<tr>
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<td>0.0</td>
<td>1.43308 63854 474</td>
<td>1.43308 63854 487</td>
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</table>
Table II. Analogous results to those of Table I, but for the complex potential $\Phi_1(z) = \phi'_1(z)$ (uniform tensile loading with $p = 2$) and its first derivative $\Phi'_1(z) = \phi''_1(z)$, computed on the basis of the boundary values $\Phi_1(t) = p/2 = 1$ of $\Phi_1(z)$, at the points $z = 0.90$ and $z = 0.99$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\Phi_1(0.90)$</th>
<th>$\Phi'_1(0.90)$</th>
<th>$\Phi_1(0.99)$</th>
<th>$\Phi'_1(0.99)$</th>
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<td>1.13155 94894 026</td>
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</tbody>
</table>

$E^*_n$ referring to the error term), but the accuracy is low because of the pole at $t = z$ when $z$ lies near the boundary $C$ of $D$, whereas for the Cauchy theorem (4), (5), etc. (but with $z \in D$ now), we are able to take into account the results of the `regularization procedure' by Ioakimidis, Papadakis and Perdios,$^{11}$ completely equivalently, we have to apply the trapezoidal quadrature rule (22) to the regularized formulae (4), (5), etc. ($z \in D$) and simply solve for $f(z)$ in (4), next for $f'(z)$ in (5), etc. The whole approach is rather simple and the final formulae for $f(z)$ and its first two derivatives (required here for the elasticity complex potentials $\Phi(z)$ and $\psi(z)$) have the forms$^{11}$

$$ f(z) \approx f_n(z) = I_{0n}(z)/J_{0n}(z), \quad (27) $$

$$ f'(z) \approx f'_n(z) = [I_{1n}(z) - f_n(z)J_{1n}(z)]/J_{0n}(z) = [I_{1n}(z)J_{0n}(z) - I_{0n}(z)J_{1n}(z)]/J_{0n}^2(z), \quad (28) $$

$$ f''(z) \approx f''_n(z) = 2[I_{2n}(z) - f_n(z)J_{2n}(z) - f'_n(z)J_{1n}(z)]/J_{0n}(z) $$

$$ \quad = 2[I_{2n}(z)J_{0n}^2(z) - I_{1n}(z)J_{0n}(z)J_{1n}(z) - I_{0n}(z)[J_{0n}(z)J_{2n}(z) - J_{1n}^2(z)]]/J_{0n}^3(z) \quad (29) $$

with $f(z)$ assumed to be an analytic function in $\bar{D} = D \cup C$ and $z \in D$. The numerical-integration related functions $I_{mn}(z)$ and $J_{mn}(z)$ appearing in the above formulae are given by$^{11}$
\[ I_{mn}(z) = \sum_{j=1}^{n} \frac{A_{jn} f(t_{jn})}{(t_{jn} - z)^{m+1}}, \quad J_{mn}(z) = \sum_{j=1}^{n} \frac{A_{jn}}{(t_{jn} - z)^{m+1}}, \quad m = 0, 1, 2, \ldots, \quad z \in D, \quad (30) \]

as is clearly verified from (2) and (3) in combination with (4), (5), etc.

In Table I we display the obtained numerical results for the aforementioned problem of a uniform tensile loading \( p = 2 \) of the elliptical region \( D \), as far as the complex potential \( \phi(z) \) and its two first derivatives are concerned, for \( z = 0.90 \) and on the basis of the exact values \( \phi(t) = \phi_1(t) = pt/2 = t \) of this potential on the contour \( C \). Both the standard approach, based on the Cauchy integral formulae (2) and (3), and the alternative approach (after the regularization), based on the Cauchy theorem (1) (further (4), (5), etc.) as well as the formulae (27) to (30) just above in this section have been used. The number of nodes \( n \) in the trapezoidal quadrature rule was selected to be \( n = 2^k \) with \( k = 2, 3, \ldots, 8 \). The numerical results in Table I show the dramatic increase of the accuracy after the regularization procedure, a rather natural outcome\(^1\) because no (pole-type) singularity is present any more. In fact, in the present application, the exact values of \( \phi(z) \) and its derivatives were obtained (after the regularization) with only \( n = 2 \) nodes!

In the same table, Table I, we also display the analogous results for a non-uniform loading of the elliptical region \( D \), selected in such a way that \( \phi(z) = \phi_2(z) = \cosh z \). (We did not pay any attention to the appropriate function \( p(t) \) in (11) neither to the related function \( \psi(z) \).) Assuming, again, the exact values of \( \phi_2(z) \) available on \( C \), we obtained the related results in Table I, which show, again, an extremely significant increase in the accuracy after the regularization and this (as was already mentioned) is completely natural.

Finally, in Table II we display analogous results for our first application (uniform loading \( p = 2 \)), but for two points \( z = 0.90 \) and \( z = 0.99 \), for \( k = 2, 3, \ldots, 10 \) and on the basis of (exact) boundary values of the derivative \( \Phi(z) = \phi'(z) = p/2 = 1 \) of the original complex potential \( \phi(z) \). Again the numerical results show the extremely significant favourable influence of the regularization procedure (which yields exact results in the present simple application) especially for \( z = 0.99 \). For this particular value of \( z, z = 0.99 \), we observe from Table II that the numerical results by the classical method (that based on the Cauchy integral formula) are essentially useless for \( n = 64 \) as far as the value of \( \Phi_1(0.99) \) is concerned and for \( n = 256 \) (or even 512) as far as the value of \( \Phi_1'(0.99) \) is concerned.

We feel that the present numerical results make it clear/verify that the Cauchy theorem should always be used instead of the Cauchy integral formula during the application of the Muskhelishvili CBIE method to the actual computation of stress and displacement components near the boundary \( C \) of the elastic region \( D \).

Obviously, quite analogous results can also be obtained in the case of infinite elastic regions \( D \) with a hole (or more than one hole or even an inclusion/inclusions), provided that the boundary values \( \phi(t) \) of the analytic function \( \phi(z) \) (which has to be computed near the boundary \( C \)) are numerically (and sufficiently accurately) available on the whole boundary \( C \).

Finally, it is clear that in the above numerical applications and in analogous applications where the trapezoidal quadrature rule for the numerical evaluation of integrals of periodic functions is employed along a closed boundary \( C \), the \( n \) functional evaluations (e.g. for the Kolosov–Muskhelishvili complex potential \( \phi(z) \) or its first derivative \( \Phi(z) = \phi'(z) \)) available for \( n \) nodes \( t_{jn} \) on the boundary \( C \) are generally useful in the case of \( 2n \) nodes \( t_{j,2n} \) too, where, in this way, only \( n \) additional functional evaluations are required.
5. THE LAURICELLA–SHERMAN COMPLEX BOUNDARY INTEGRAL EQUATION

Traditionally, much more popular than the Muskhelishvili CBIE (complex boundary integral equation) has been the so-called Lauricella–Sherman (or Sherman–Lauricella) CBIE, actually derived by Sherman in complex form\textsuperscript{25–30} (see, e.g., the monograph by Muskhelishvili\textsuperscript{12}) during the period 1940–43. The Lauricella–Sherman CBIE is also based on the boundary condition (11) (valid on the boundary $C$ of the elastic region $D$) and it has the following form:\textsuperscript{12}

$$k\omega(t) + \frac{k}{2\pi i} \oint_C \omega(\tau) \, d\log \frac{\tau-t}{\tau-\tau} - \frac{1}{2\pi i} \oint_C \omega(\tau) \, d\frac{\tau-t}{\tau-\tau} = p(t), \quad \tau \in C,$$

(31)

where, in the right-hand side, $p(t)$, a non-analytic function, is defined exactly as in Section 3 and $\omega(t)$ is the unknown (and also non-analytic) density function on the boundary $C$. A comparison between the CBIEs (12) and (31) reveals no essential difference beyond a sign in the left-hand sides and, further, the right-hand sides (which are known functions). In fact, the right-hand side has to be computed in the Muskhelishvili CBIE (12) (by using (13)) on the basis of $p(t)$, whereas it coincides with $p(t)$ in the Lauricella–Sherman CBIE (31) although an indefinite integration of the tractions function (on $C$) is still required (for the determination of $p(t)$). Yet, the whole situation becomes much more complicated when the complex potentials $\phi(z)$ and $\psi(z)$ themselves have to be computed (in the case of the Lauricella–Sherman CBIE) and we feel that the appearance of the present (separate) section of the paper is completely justified.

More explicitly, the major difference between the Muskhelishvili and the Lauricella–Sherman CBIEs consists in the fact that in the former CBIE the densities in the Cauchy-type integrals defining the complex potentials $\phi(z)$ and $\psi(z)$ are just the boundary values of these potentials (this is clear from (14) and (15)), but this completely ceases being the case in the latter CBIE. In fact, the fundamental formulae for the approximate computation of these potentials get the following forms in the Lauricella–Sherman CBIE:\textsuperscript{12}

$$\tilde{\phi}(z) = \frac{1}{2\pi i} \oint_C \frac{\tilde{\omega}(t)}{t-z} \, dt, \quad \tilde{\phi}'(z) = \frac{1}{2\pi i} \oint_C \frac{\tilde{\omega}(t)}{(t-z)^2} \, dt, \quad \tilde{\phi}''(z) = \frac{1}{\pi i} \oint_C \frac{\tilde{\omega}(t)}{(t-z)^3} \, dt, \quad z \in D,$$

(32)

as far as $\phi(z)$ is concerned (where, again, we took the liberty to add the expressions for the required first two derivatives of $\phi(z)$ analogously to what was already done in (14))\textsuperscript{12}

$$\tilde{\psi}(z) = \frac{k}{2\pi i} \oint_C \frac{\tilde{\omega}(t)}{t-z} \, dt - \frac{1}{2\pi i} \oint_C \frac{\tilde{\omega}'(t)}{t-z} \, dt, \quad z \in D,$$

(33)

as far as $\psi(z)$ is concerned or, probably better,\textsuperscript{12}

$$\tilde{\psi}(z) = \frac{k}{2\pi i} \oint_C \frac{\tilde{\omega}(t)}{t-z} \, dt + \frac{1}{2\pi i} \oint_C \frac{\tilde{\omega}(t)}{t-z} \, dt - \frac{1}{2\pi i} \oint_C \frac{\tilde{\omega}(t)}{(t-z)^2} \, dt, \quad z \in D,$$

(34)

the formulae analogous to (but somewhat simpler than) (17) and (18), respectively. The derivative $\psi'(z)$ of $\psi(z)$ is also required for the subsequent computation of the stress components as is clear from (8) and, for this task, a differentiation of (34) is sufficient.

In the above (approximate) expressions for the complex potentials $\phi(z)$ and $\psi(z)$ (and their first derivatives required in (6) to (8)), we implicitly assumed that we have already numerically solved the fundamental Lauricella–Sherman CBIE (31) and, therefore, we have available its approximate solution.
STRESS COMPONENTS NEAR A BOUNDARY

\[ \tilde{\omega}(t) \approx \omega(t) \], further leading to approximate expressions \((\tilde{\phi}(z), \tilde{\psi}(z), \text{etc.})\) for the complex potentials and their derivatives. This is natural.

The difficulty here, just from the computational point of view, simply lies in the fact that the unknown function (the density function) \(\omega(t)\) (better its aforementioned approximation \(\tilde{\omega}(t)\)) is no more the boundary value of an analytic function (in the Lauricella–Sherman CBIE (31)) contrary to what has been the case in the much less popular Muskhelishvili CBIE (12). The consequence of this situation is simply that the computational approach already described and (rather efficiently) numerically illustrated in the previous section is just completely inapplicable to the Lauricella–Sherman method adopted in this section. Evidently, the remedy for this inconvenient situation is the appropriate generalization of the approach of the previous section so that it can become applicable to Cauchy-type integrals (and their derivatives) with a non-analytic (a general) density function, here \(\tilde{\omega}(t)\). This task will be undertaken in the next section and, in our opinion, it constitutes a non-trivial (although conceptually not very difficult) generalization of the results of Section 4, which were based just on the use of the Cauchy theorem (after a regularization of the Cauchy integral formula) as was already originally suggested by Ioakimidis, Papadakis and Perdios\(^\text{11}\) from the numerical-analysis point of view.

6. NON-ANALYTIC DENSITY FUNCTIONS

In this section we will generalize the results of Section 4 to non-analytic density functions exactly as is the case in the Lauricella–Sherman CBIE (31) in the previous section. More explicitly, we can consider the complex potentials \(\phi(z)\) (and its derivatives \(\Phi(z) = \phi'(z)\) and \(\Phi'(z) = \phi''(z)\)) and \(\psi(z)\) in (32) and (34), respectively (for their present approximations).

The idea in the approach of this section is extremely simple: in order to compute the value of an analytic function \(f(z)\) at a point \(z\) inside the elastic region \(D\) (but close to its boundary \(C\)) by using the Cauchy-type integral (10) with \(g(t)\) non-analytic there exactly as is the case for the densities of both complex potentials \(\phi(z)\) and \(\psi(z)\) in the Lauricella–Sherman method of the previous section, we need just to use the approach of Section 4 again, but after the numerical computation of the boundary values \(f(t)\) of \(f(z)\) on the basis of the available values of the density \(g(t)\) on \(C\), which, obviously, is defined only on the closed boundary \(C\) and neither inside nor outside this boundary.

This task can be easily numerically performed by using the very well-known Sokhotski–Plemelj formulae. In our case (of a finite elastic medium \(D\) bounded by the closed contour \(C\)), we simply have\(^\text{12,22}\)

\[ f(t) = \frac{1}{\pi} g(t) + \frac{1}{2\pi i} \int_C \frac{g(\tau)}{\tau - t} \, d\tau, \quad f(t) \equiv f^+(t), \quad t \in C, \tag{35} \]

(on the basis of (10) and with the Cauchy-type integral defined in the principal value sense).

Therefore, our sole task is just to compute \(f(t)\), by using the above formula, at the required nodes \(t_{jn}\) of the quadrature rule (22) (e.g., in Section 4 for an elliptical contour, the nodes \(t_{jn}\) are defined by (23) and (24)) and that’s all. The first (the non-integral) term in the right hand-side of (35) is directly available (at least at the nodes \(t_{jn}\) to be used) from the numerical solution of the CBIE (e.g. the Lauricella–Sherman CBIE (31)), whereas the second (the integral) term in (35) can be directly computed by using an appropriate quadrature rule for Cauchy-type, but now principal value, integrals. This is a very easy task, since Cauchy-type principal value integrals can be numerically evaluated very accurately because no pole of the integrand is present near the integration contour (here the boundary \(C\), whereas the pole \(\tau = t\) can be efficiently taken into account.

In fact, the principle of this computation is, essentially, the singularity subtraction, e.g., as far as
our fundamental Sokhotski–Plemelj formula (35) is concerned, we can rewrite it as

\[ f(t) = \frac{1}{2} g(t) + \frac{1}{2\pi i} \int_C \frac{g(\tau) - g(t)}{\tau - t} \, d\tau + \frac{g(t)}{2\pi i} \int_C \frac{1}{\tau - t} \, d\tau, \quad t \in C, \tag{36} \]

or, better, by using the Cauchy integral formula (2) for \( f(z) = 1 \) as \( z \) tends to a point \( t \) of \( C \) and with the Sokhotski–Plemelj formula (35) taken again into account, it is directly seen that

\[ \frac{1}{2\pi i} \int_C \frac{1}{\tau - t} \, d\tau = \frac{1}{2}, \quad t \in C. \tag{37} \]

Therefore, (36) takes the final form

\[ f(t) = g(t) + \frac{1}{2\pi i} \int_C \frac{g(\tau) - g(t)}{\tau - t} \, d\tau, \quad t \in C, \tag{38} \]

containing only an ordinary contour integral and not requiring any special quadrature rule (i.e. for Cauchy-type principal value integrals): just an interpolatory quadrature rule for ordinary integrals of the form (22) is absolutely sufficient.

If one insists on using a quadrature rule for Cauchy-type principal value integrals in (35), this is also a very easy task and several references are available in the literature. For example, for the trapezoidal quadrature rule, which was already employed in Section 4, we can directly use the results by Chawla and Ramakrishnan,\textsuperscript{69} exactly as was already done by the author long ago\textsuperscript{35}, by Theotokoglou\textsuperscript{10} and by other researchers mainly in the classical theory of elasticity for the CBIE method. These results\textsuperscript{69} reveal that for this particular (the trapezoidal) quadrature rule, its original expression (22) remains valid for Cauchy-type principal value integrals (such as that in the right-hand side of (35)) provided that the real parameter \( \theta \) in the complex free variable \( t \) (along \( C \)) in (19) is restricted to the values

\[ \gamma_{kn} = 2(2k - 1)\pi/(2n), \quad k = 1, 2, \ldots, n, \tag{39} \]

i.e. just to the midpoints of the values \( \theta_{jn} \) in (23) for \( \theta \) corresponding to the nodes \( t_{jn} \) in the quadrature rule (22). Incidentally, a 'phase' angle \( \alpha \) (an arbitrary real constant) can be added to both (23) and (39).

If this is not the case, an additional term (proportional to \( g(t) \)) has to be added to the quadrature rule (22).\textsuperscript{69,10} Yet, in our personal opinion, from the practical point of view, the preferable possibility is to use just (38) for essentially any value of \( t \) i.e. with the exception only of the nodes \( t_{jn} \) in (22). For these nodes, \( t = t_{jn}, j = 1, 2, \ldots, n \), it is clear that (22) is still applicable to the numerical evaluation of the ordinary integral in (38), but the approximate expression to this integral now contains the derivative \( g'(t_{jn}) \) for this particular node and this is somewhat impractical. We can also add that for a general interpolatory (not just the trapezoidal) quadrature rule the use of (38) seems to be the best possibility; otherwise, appropriate nodes (such as those corresponding to (39) in the trapezoidal quadrature rule) should have been determined in advance.

In any case, it can also be mentioned that in (35) or, better, (38) generally no derivative of \( g(t) \) needs to be computed and it is completely reasonable to assume that, in this way, once the density \( g(t) \) is available on the contour \( C \) (e.g. \( \omega(t) \approx \tilde{\omega}(t) \) in the Lauricella–Sherman method of Section 5 from the numerical solution of the related CBIE (31)), then the boundary values \( f(t) \) of \( f(z) \) (in our case from the interior \( D \) of \( C \), \( f(t) \equiv f^+(t) \), as \( z \to t \in C \)) are also available (e.g. \( \phi(t), t \in C \), in the Lauricella–Sherman method). Therefore, the numerical approach already illustrated in Section 4 and permitting us to get rid of the pole \( z \) in the Cauchy-type integrals and their derivatives at a point \( z \) near the boundary \( C \) of the elastic region \( D \) is still completely applicable here (essentially without
Table III. Analogous results to those of Table II, but for the complex potential $\phi(z)$ and its first derivative $\Phi(z) = \phi'(z)$ on the basis of the non-analytic density function $\omega(t) = |t|$ ($t \in C$) in the Lauricella–Sherman CBIE (31) and again at the points $z = 0.90$ and $z = 0.99$. The computations in columns 3 and 5 were based on the use of (35)

<table>
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<tr>
<th>$n$</th>
<th>$\phi(0.90)$</th>
<th>$\hat{\phi}(0.90)$</th>
<th>$\phi(0.99)$</th>
<th>$\hat{\phi}(0.99)$</th>
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</tr>
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<td>3.48432 55002 145</td>
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<tr>
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<td>0.86278 99415 202</td>
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</tr>
<tr>
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<td>0.86278 99415 202</td>
<td>0.97444 52137 651</td>
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</table>

<table>
<thead>
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<th>$\Phi(0.90) = \hat{\Phi}'(0.90)$</th>
<th>$\Phi(0.99) = \hat{\Phi}'(0.99)$</th>
<th>$\Phi(0.99) = \hat{\Phi}'(0.99)$</th>
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</thead>
<tbody>
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<td>4</td>
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<td>0.35121 95121 951</td>
<td>1249.80486 86384 415</td>
<td>0.38634 14634 146</td>
</tr>
<tr>
<td>8</td>
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<td>0.35148 62356 068</td>
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<td>0.35099 07239 972</td>
<td>0.40261 42523 854</td>
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<td>0.35099 07239 972</td>
<td>0.37179 46718 658</td>
<td>0.37179 28922 617</td>
</tr>
</tbody>
</table>

any serious loss of accuracy) in spite of the fact that $g(t)$ ($\omega(t)$ in the Lauricella–Sherman method) is a non-analytic function.

The present approach, based on the use of the Sokhotski–Plemelj formula (35), was applied to the case of the non-analytic density function

$$\omega(t) = \sqrt{tt} = |t|, \quad t \in C,$$

(40)

for the computation of the complex potential $\phi(z)$ and its first derivative $\Phi(z) = \phi'(z)$ in the plane elasticity problem of an elliptical region $D$ bounded by a contour $C$ exactly as in Section 4. The above non-analytic density function $\omega(t)$ was assumed to be available in advance from the solution of the related CBIE (e.g. (31) in the Lauricella–Sherman method). Here our attention will be restricted just to the efficient computation of $\phi(z)$ and $\Phi(z) = \phi'(z)$ at a point $z$ near $C$ exactly as was already done in Section 4, but just for analytic density functions there (more explicitly, under the direct availability of the boundary values $\phi(t)$ of $\phi(z)$ itself).

In Table III we display the derived numerical results for $z = 0.90$ as well as $z = 0.99$ and $n = 2^k$
Table IV. Comparison of the numerical results of Table III (again for the complex potential \( \phi(z) \) and its first derivative \( \Phi(z) = \phi'(z) \) at \( z = 0.99 \) on the basis of the non-analytic density function \( \omega(t) = |t|, \ t \in C \) with the computations based now on the use of either (35) (columns 2 and 4, values with a hat) or (38) (columns 3 and 5, values with a check, new values)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \hat{\phi}(0.99) )</th>
<th>( \tilde{\phi}(0.99) )</th>
<th>( \hat{\Phi}(0.99) = \hat{\phi}'(0.99) )</th>
<th>( \tilde{\Phi}(0.99) = \tilde{\phi}'(0.99) )</th>
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<td>64</td>
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<td>0.89533 28000 57075</td>
<td>0.37179 28922 61684</td>
<td>0.37179 28922 61684</td>
</tr>
</tbody>
</table>

\( (k = 2, 3, \ldots, 10) \) by the present approach (again with \( a = 1 \) and \( b = 1/2 \)) for the above non-analytic density function \( \omega(t) = |t| \) and by using the trapezoidal quadrature rule again (exactly as in Section 4) and both (i) the ‘naive’ approach (no care about the first-order pole of \( \phi(z) \) and the second-order pole of \( \Phi(z) = \phi'(z) \) at \( z \)) and (ii) the first approach already described in this section (based on the use of the Sokhotski–Plemelj formula (35)) although (38) could also, perfectly, have been used instead as will be really done below. The same nodes \( t_{jn} \) in (24) (with the values \( \theta_{jn} \) of the parameter \( \theta \) determined from (23)) were employed during the numerical computation of \( \phi(t) \) by using (35) and the boundary point \( t \) was restricted to the values \( t_{kn}^* \) determined from

\[
t_{kn}^* = a \cos \gamma_{kn} + ib \sin \gamma_{kn}, \quad k = 1, 2, \ldots, n, \tag{41}
\]

where the parametric values \( \gamma_{kn} \) are defined by (39). Evidently, this assures the legitimacy of directly using the trapezoidal quadrature rule for ordinary integrals not only in (38) (surely directly valid for almost any value of \( r \)), but also in (35). It is understood that the trapezoidal quadrature rule in Section 4 will now have the points \( t_{kn}^* \) (instead of \( t_{jn} \) there) as nodes, since the latter nodes, \( t_{jn} \), were already used in (35) or, possibly, equivalently, (38). Because of the periodicity of \( \omega(t) \) along \( C \) (with respect to the parameter \( \theta \in [0, 2\pi) \)), this slight modification has no essential influence on the whole approach: it is just a small change of ‘phase’ in the nodal values of \( \theta \) without any serious consequence. Moreover, from the numerical results of Table III the dramatic increase of the accuracy by applying the approach of the present section (especially for \( z = 0.99 \)) is clearly observed.

Finally, it seems to be of some interest to compare the numerical results for \( \phi(z) \) and \( \Phi(z) = \phi'(z) \) obtained by using the approach of this section and based on the use of (i) (35) (with a Cauchy-type principal value integral) and (ii) (38) (with an ordinary integral instead after a regularization), the latter formula resulting from the former one with the help of (36) and (37) as was already mentioned. This is made in Table IV (for \( z = 0.99 \) only and on the basis of the already used nodes \( t_{jn} \) and \( t_{kn}^* \) for \( n = 2^k \ (k = 2, 3, \ldots, 6) \). From the numerical results of Table IV it becomes clear that there is no significant difference in the accuracy by using either (35) (values with a hat, already displayed in Table III, but with a slightly lower number of digits there) or (38) (values with a check, new values). In fact, the numerical results for the complex potential \( \phi(z) \) (with 15 displayed decimal digits) coincide in Table IV for \( n = 32, 64 \), quite similar being the case for the derivative \( \Phi(z) = \phi'(z) \) but for \( n = 64 \) (and to 14 decimal digits for \( n = 32 \).
7. SINGULAR/HYPERSINGULAR COMPLEX BOUN DARY INTEGRAL EQUATIONS

Beyond the Muskhelishvili and the Lauricella-Sherman (almost) Fredholm CBIEs (complex boundary integral equations), singular and hypersingular CBIEs are also popular in classical plane elasticity problems, the latter (the hypersingular) during the last few years. Some related references were also given in Section 1. In all cases, the Kolosov-Muskhelishvili complex potentials and their necessary derivatives (as far the computation of the stress and displacement components is concerned) are expressed in terms of appropriate Cauchy-type integrals of the form (10) and their derivatives with non-analytic density functions \( g(t) \), which are frequently interpreted in terms of edge dislocation densities or even concentrated force densities. In all of these cases, the results of the previous section are directly applicable for a much more accurate numerical computation of the stress and displacement components, but after the actual numerical solution of the related CBIE, i.e. the approximate computation of the unknown and non-analytic density function \( g(t) \) in (10) on the boundary \( C \) of the elastic region \( D \).

For example, the author considered the following Cauchy-type expressions for the first derivatives \( \Phi(z) \) and \( \Psi(z) \) of the Kolosov-Muskhelishvili complex potential \( \phi(z) \) and \( \psi(z) \): \( ^{32,35,33,38} \)

\[
\Phi(z) = \frac{1}{2\pi i} \oint_C \frac{g(t)}{t-z} dt, \quad \Psi(z) = \frac{1}{2\pi i} \oint_C \frac{h(t)}{t-z} dt, \quad z \in D, \tag{42}
\]

(with \( g(t) \) and \( h(t) \) denoting, in general, non-analytic density functions, defined on \( C \) only) together with the appropriate physical (equilibrium) and uniqueness conditions supplementing the fundamental traction-based condition \( ^{32,35,33,38} \)

\[
\Phi(t) + \Phi(t) + \frac{dt}{dt} [\bar{\Phi}'(t) + \Psi(t)] = \sigma_n(t) - i\sigma_t(t), \quad t \in C, \tag{43}
\]

where \( \sigma_n(t) \) and \( \sigma_t(t) \) denote the normal and shear (respectively) traction components along the boundary \( C \) of the elastic medium \( D \) for convenience assumed a bounded simply-connected region.

Then the unknown non-analytic density function \( h(t) \) (concerning \( \Psi(z) \)) can be determined (under appropriate assumptions concerning the boundary conditions for the infinite region \( E \) outside \( D \) as well) from (43) on the basis of the equally unknown and non-analytic density function \( g(t) \) (concerning \( \Phi(z) \)) and the known tractions \( \sigma_{n,t}(t) \) and, therefore, the original expression for \( \Psi(z) \) in (42) takes the final form \( ^{32,35,33,38,16} \)

\[
\Psi(z) = -\frac{1}{2\pi i} \oint_C \frac{g(t)}{t-z} dt - \frac{1}{2\pi i} \oint_C \frac{\bar{g}(t)}{(t-z)^2} dt, \quad z \in D. \tag{44}
\]

Finally, the following CSIE (Cauchy-type singular integral equation) is obtained \( ^{32,35} \)

\[
\Re \left[ \frac{1}{\pi i} \oint_C \frac{g(\tau)}{\tau-t} d\tau \right] - \frac{dt}{dt} \left[ \frac{1}{2\pi i} \oint_C \frac{g(\tau)}{\tau-t} d\tau + \frac{1}{2\pi i} \oint_C \frac{\bar{\tau}-\bar{\tau}}{(\tau-t)^2} g(\tau) d\tau \right] = \sigma_n(t) - i\sigma_t(t), \quad t \in C. \tag{45}
\]

(A slightly more complicated CSIE/CBIE was also obtained in the case of an inclusion \( ^{33} \) and in several similar problems in plane elasticity.) After the efficient numerical solution of the CSIE/CBIE (45) and the approximate determination of \( g(t) \), we can proceed to the subsequent determination of the stress components by using the fundamental Kolosov-Muskhelishvili formulae (7) and (8) with the complex potentials there computed with the help of the first of (42) and (44) (as well as their first derivatives).

Alternatively, we can base our analysis on the original complex potentials \( \phi(z) \) and \( \psi(z) \) (instead of their first derivatives \( \Phi(z) \) and \( \Psi(z) \) used just above) by employing (32) and (34) (as well as the
first derivative of the latter equation). Then, essentially, the first of (42) is substituted by the first of (32) (since $\Phi(z) = \phi'(z)$) and, therefore, the order of the pole in the Cauchy-type integrals for $\Phi(z)$ and $\Phi'(z)$ increases by one, completely analogous being the case with $\Psi(z)$. The outcome of this situation is the derivation of a HSIE (Hadamard–Mangler-type hypersingular integral equation) as our fundamental CBIE instead of a CSIE (Cauchy-type singular integral equation), as has been the case with (45), but, obviously, the whole numerical approach in the previous section remains intact although, it should be confessed, third-order poles appear as well whenever we insist to use HSIEs as our CBIEs. We do not feel that it is necessary to display further details on analogous CBIEs for plane elasticity problems (possibly including holes, inclusions, etc.).

A very large number of related papers, leading either to (almost) Fredholm-type or to Cauchy-type or even to Hadamard–Mangler-type (i.e. regular or singular or even hypersingular, respectively) CBIEs, is available in the literature (some references were already provided in Section 1) and the interested reader can choose the method that is more convenient to him from the physical–mechanical, the analytical or even the numerical point of view. (For example, nobody will disagree that the appearance of a higher-order pole in the Cauchy-type integrals is not helpful for the more accurate computation of the related integrals: either principal-value singular/finite-part hypersingular on the contour $C$ or just nearly singular/hypersingular in the close vicinity of the contour $C$, the latter being the sole question that we try to study in the present particular paper, of course, inside the convenient and interesting framework offered by the classical, complex-variable-based theory of static plane isotropic linear elasticity and the related CBIEs, some of which were already reported in Sections 3 and 5 as well as in the present section.

8. CONCLUSIONS–DISCUSSION–GENERALIZATIONS

From the above results we are also led to the following conclusions/generalizations/comments:

1. Independently of the form of the CBIE: almost regular (Fredholm-type), singular (Cauchy-type) or hypersingular (Hadamard–Mangler-type) in a plane elasticity problem related to an elastic region $D$ bounded by a closed contour $C$, there is no difficulty for the very accurate computation of the stress components $\sigma_x$, $\sigma_y$ and $\sigma_{xy}$ at points $z$ near the boundary $C$ of the region $D$ with essentially the same accuracy to that achieved for points $z$ of $D$ far away $C$, provided that we use the Cauchy theorem (instead of the much more obvious Cauchy integral formula), thus performing a kind of regularization before the actual computation of the related complex contour integrals. In this approach the use of just an ordinary quadrature rule (such as the trapezoidal quadrature rule used here) seems to be completely sufficient.

2. The above approach holds true either for analytic density functions in the Cauchy-type integrals (i.e. the boundary values of the complex potentials themselves) or for non-analytic (and much more popular in practice) density function such as those associated with the Lauricella–Sherman CBIE and the probably even more popular CBIEs based on dislocation densities under traction boundary conditions (and traction densities under displacement boundary conditions) on $C$ although, it must be confessed, in the case of non-analytic density functions the computational cost (aiming at the aforementioned regularization) is significantly increased, but by no means is it prohibitive as was already observed.

3. A rather interesting possibility (at least from the theoretical point of view) is the generalization of the results in Reference 11 as well as the present results to the case of analytic functions of quaternions $q = t + ix + jy + kz = (t, x, y, z)$ instead just of complex variables $z = x + iy = (x, y)$. (An interesting and well-written related review paper is that by Deavours.75) The related generalization of the Cauchy integral formula is called the Cauchy–Fueter integral formula75
and it is reasonable to assume that this generalization does not present any essential difficulty from the theoretical point of view although it is understood that the actual computational effort (during the integrations) will be significantly increased because of the fact that in the quaternion calculus we have four real variables:  \( t, x, y \) and \( z \) (the last three, \( x, y \) and \( z \), frequently referring to the Cartesian coordinates in space and the first one, \( t \), being proportional to time). Yet, for quaternions, much more important is their appropriate use in engineering mechanics and, particularly, elasticity problems. The interested reader can consult, e.g., the papers by Geradin and Cardona,\(^{76}\) Pimenov and Pushkarev\(^{77}\) and Kutrunov.\(^{78}\)

4. The above results for plane isotropic elasticity are also directly applicable to antiplane elasticity, anisotropic plane and antiplane elasticity, classical plane fluid dynamics and potential theory and, in general, problems where complex potentials/analytic functions are used in combination with CBIEs along closed contours \( C \).

5. It seems really unfortunate, but the author failed to generalize the present results to crack problems (or to analogous problems such as line inclusion problems) in plane/antiplane elasticity, where an open contour \( L \) is present (instead of a closed contour \( C \) above). This happened since the Cauchy theorem (1) is inapplicable to open contours \( L \). On the other hand, by no means can this author recommend the use of the conformal mapping from a single crack \( L \) to the interior (or exterior) \( D \) of a circle, etc. because this approach is extremely impractical. Therefore, the aforementioned generalization (if possible) remains an open problem for the interested reader of the paper.

6. On the contrary, the above results are also applicable to the case of regions \( D \) with a hole/holes and/or an inclusion/inclusions, i.e. to cases where \( C \) is includes an interior boundary/boundaries of a finite/infinite elastic region \( D \), simply since the Cauchy theorem (1) and the other related fundamental formulae (the Cauchy integral formula (2) even the Sokhotski–Plemelj formula (35)) essentially remain holding true in this general case\(^{1,12,22}\) exactly as is also the case for the related CBIEs after slight modifications/generalizations mainly concerning the uniqueness and the single-valuedness of their solutions and, in some cases, supplemented by additional integral conditions (single-valuedness conditions).\(^{12}\) For example, the fundamental Cauchy integral formula (2) takes the following well-known form:\(^{12}\)

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} \, dt + f(\infty), \quad z \in D, \tag{46}
\]

for an infinite elastic medium \( D \equiv D^- \) (with an internal boundary \( C \), a hole) under the assumption that \( f(z) \) is analytic in \( D \), including the point at infinity, and continuous in \( \overline{D} = D \cup C \).\(^{12}\)

7. As was already mentioned, this author is not an expert in the BEM (boundary element method) and this is a serious reason that no attempt was made above for the extension of the present results to the BEM. Yet, the BEM is also based on BIEs (boundary integral equations) and the CVBEM on CBIEs (complex boundary integral equations), but the quadrature rules used there are not interpolatory quadrature rules: they are based just on boundary elements along the contour \( C \). In any case, it seems that the above results are, in principle, applicable to the BEM although their actual–practical related offer (if of some interest) cannot be estimated here because of the existence of a vast literature on the efficient computation of the stress and displacement components near a boundary \( C \) by using the BEM (see, e.g., References 3 to 9), which has to be studied in detail before any clear conclusions on the possible usefulness of the present approach to the BEM can be reached. Therefore, in this paragraph we will restrict our attention just to the principle of using the above results in the BEM, again for classical
stress components near a boundary

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More explicitly, we assume that the BEM has been already applied to a concrete problem and the related numerical solution is available. The original boundary conditions and this solution permit us to have available both the traction components \( \sigma_n \) and \( \sigma_t \) and the displacement components \( u \) and \( v \) on the boundary \( C \) of the region \( D \). Of course, the BEM does not use complex variables (although the CVBEM does), but this does not constitute a serious obstacle to the present approach, which, nevertheless, is exclusively based on complex variables. More explicitly, after the numerical solution of the fundamental BIE in the BEM (for plane elasticity by using boundary elements; see, e.g., Reference 79) we have available both the traction and the displacement components along the boundary \( C \). Next we can use the fundamental Kolosov-Muskhelishvili formulae (6) to (8) for the ‘interpretation’ of our numerical results in terms of the complex potentials \( \phi(z) \) and \( \psi(z) \) and/or their appropriate derivatives so that the approach of the present paper can become really applicable to the BEM. For example, by differentiating the complex conjugate of (6) with respect to \( \bar{t} \), we easily get

\[
\Phi(t) - \kappa \overline{\Phi(t)} + \frac{d}{dt} [\bar{t}\Phi'(t) + \Psi(t)] = -2\mu \left( \frac{du}{dt} - i\frac{dv}{dt} \right), \quad t \in C. \tag{47}
\]

Next, by combining this formula with (43) (under the assumption that the right-hand side in (47) can be easily computed on the basis of the available displacement components \( u \) and \( v \) on the boundary \( C \) after an appropriate differentiation, the right-hand side in (43) being directly available from the boundary conditions or from the BEM solution), we can easily determine (through a simple subtraction) \( \Phi(\bar{t}) \), completely equivalently, its complex conjugate \( \Phi(t) \), on \( C \). Then the approach of Section 4 above is directly applicable. In an analogous way, we can next proceed with \( \Psi(t) \) by using either (47) or, perhaps better, (43). Evidently, it is understood that this is an extremely elementary approach and, therefore, improvements may be possible.

8. Moreover, what may be even more important from the practical point of view is the substitution of the above-described complex-variable approach, based on analytic functions in the complex plane, by a corresponding real-variable approach (with \( x \) and \( y \) used distinctly instead of their combination \( z = x + iy \)), based just on harmonic functions (possibly, also on biharmonic functions), since both the real and the imaginary part of an analytic function are harmonic functions.\(^1\) We are unaware whether and (in an affirmative case) to which extent this task has been undertaken in the past, but such an approach (based just on real variables) may look much more familiar to the researchers in the BEM and the simple users of the BEM. The idea is very simple: just an attempt to ‘translate’ the approach of the present paper to real variables (by using only harmonic functions). Is this a feasible task mainly from the practical point of view (i.e. to reach very accurate results for the stress components near a boundary \( C \) by using the BEM, where nearly singular integrals are generally present) in combination with the present method? (Incidentally, we can add that a ‘translation’ of a complex-variable approach to real variables was made by the author in the completely different computational environment of the numerical determination of the roots of two real nonlinear equations in two unknowns with an extension, by using quaternions, to four real nonlinear equations.\(^8\)) In this author’s opinion, a reconciliation between the present results and those in the BEM looks a very interesting possibility and, moreover, this author cannot accept in advance that this (seemingly reasonable) approach is just impossible. Of course, the situation is much more comfortable with the CVBEM, since, actually, in the CVBEM we just use boundary elements in the CBIEs instead of ordinary interpolatory quadrature rules (as was done in this paper with the use of the trapezoidal quadrature rule), but the whole approach is based again on analytic functions (including Cauchy-type integrals).
REFERENCES


