Remarks on the Gauss quadrature rule for a particular class of finite-part integrals

Nikolaos I. Ioakimidis

Division of Applied Mathematics and Mechanics, Department of Engineering Sciences, School of Engineering, University of Patras, GR-265 04 Patras, Greece

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REMARKS ON THE GAUSS QUADRATURE RULE
FOR A PARTICULAR CLASS OF FINITE-PART INTEGRALS

NIKOLAOS I. IOAKIMIDIS

Correspondence address: Prof. N. I. Ioakimidis
P. O. Box 1120
GR-261.10 Patras
GREECE

Telephone numbers: (Greece)+61+432-257,
(Greece)+61+997-378,
(Greece)+61+310-240

Telefax number: (Greece)+61+310-240

Division of Applied Mathematics and Mechanics, School of Engineering,
University of Patras, P. O. Box 1120, GR-261.10 Patras, Greece

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SUMMARY

The method of finite-part (Hadamard-type or hypersingular) integrals is already a standard approach in applied numerical methods in engineering and, mainly, in acoustics, fluid dynamics, elasticity and fracture mechanics. Here we study the convergence of the nonclassical (possessing a negative node, outside the integration interval, as well as a negative weight corresponding to this node) Gauss–Legendre quadrature rule for finite-part integrals on the interval [0,1] with the weight function 1/y and we prove its validity for integrands possessing a continuous first derivative. The case of integrands possessing a higher-order derivative is also considered and the rate of convergence is established. These results are based on the derivation of a bound for the negative node in the above quadrature rule and, further, on the theory of convergence of the classical Gauss quadrature rule. Numerical results, verifying the results of this paper, are also displayed. The present results can be generalized to the Gauss–Laguerre similarly nonclassical quadrature rule for finite-part integrals (on the interval [0, \infty)) as well as to the two special, but important in engineering applications, nonclassical Gaussian quadrature rules for Cauchy-type principal-value integrals and Mangler-type finite-part integrals having been proposed by Tsamasphyros and Dimou.
1. INTRODUCTION

Singular integral equations with finite-part (Hadamard-type or hypersingular or strongly singular and, in a special case of a second-order singularity, Mangler-type) integrals (hypersingular integral equations, HSIEs) play today a very important role in applied mechanics (two- and three-dimensional elasticity including fracture mechanics, elastic-wave scattering, etc.) and engineering in general (including, of course, acoustics, fluid dynamics, etc.). The pioneering results on the numerical evaluation of this class of integrals (still referenced quite frequently in recent related publications) are due (as is very well known) to Kutt.\textsuperscript{1-4} In fact, in 1975 Kutt published in two special reports his results on the construction of quadrature rules (both with equispaced nodes and Gaussian) for the numerical evaluation of finite-part (hypersingular) integrals on a finite interval, directly reducible to $[0,1]$. The first report\textsuperscript{1} presents a summary of the results for these quadrature rules and, mainly, tables of nodes and weights. The second report (the Ph.D. thesis of Kutt)\textsuperscript{2} presents a wide series of theoretical results for finite-part integrals and proceeds to the construction of the aforementioned quadrature rules (both with equispaced nodes and Gaussian). An equispaced quadrature rule for finite-part integrals involving a logarithmic singularity was also derived by Kutt in 1977 in another special report.\textsuperscript{3} Only a very short paper on the results of References 1 and 2 was published.\textsuperscript{4} Yet, these (and additional) reports have been easily accessible both from the Nasionale Navorsingsinstituut vir Wiskundige Wetenskappe—National Research Institute for Mathematical Sciences in South Africa (where they were published) and from the National Technical Information Services in U.S.A. (where copies are available). We assume here that these important results are accessible to the reader and this has been already the case with many authors having been interested and really making reference to Kutt's results.

Here we will restrict ourselves just to one special class of finite-part integrals, those with the weight function $1/y$ on the fundamental interval $[0,1]$ of the form

$$I = \int_0^1 \frac{f(y)}{y} \, dy,$$

where the integrand $f(y)$ is assumed possessing a continuous first derivative on $[0,1]$ or, at least, in a neighbourhood of $y = 0$. This is the simplest finite-part integral, but also the most important from the practical point of view. From the classical definition of finite-part integrals of the form (1.1),\textsuperscript{2} $I$ can be written as an ordinary integral too, namely,

$$I = \int_0^1 \frac{f(y) - f(0)}{y} \, dy.$$

The application of classical quadrature rules (like the Gauss quadrature rule) to (1.2) is completely possible. The resulting rules from open interpolatory quadrature rules on $[0,1]$ for ordinary integrals become, obviously, semi-closed rules for the finite-part integral (1.1) (because of (1.2)). More specifically, the classical Gauss quadrature rule\textsuperscript{6-8} applied to (1.2) yields the Radau quadrature rule, with the preassigned node $y = 0$, for (1.1).\textsuperscript{2}

Before proceeding to our results (already mentioned in the summary of this paper) in the subsequent four sections, we wish to explain the importance of integrals of the form (1.1). At first, as was already proposed by Kutt in 1975,\textsuperscript{4} quadrature rules for the numerical evaluation of $I$ in (1.1) permit the evaluation of one-dimensional Cauchy-type principal value integrals. This class of integrals appears very frequently in a lot of branches of engineering including (but not restricted at all to) acoustics, fluid dynamics, elasticity and fracture mechanics. This remark was
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further considered in some detail by Brebbia, Telles and Wrobel, where the table of Kutt for the Gauss quadrature rule for \( f \) (nodes and weights) was reproduced for \( n = 2, 3, \ldots, 8 \) nodes with an accuracy of 30 significant digits and, further, really used in practice in several references. (In passing, we notice that the results of Kutt are also briefly mentioned in Reference 6.)

Beyond this application (where alternative techniques are also well-established and, frequently, more powerful), the case of two-dimensional Cauchy-type principal value integrals, appearing also in the same branches of engineering, almost suggests the use of one-dimensional finite-part integrals of the form (1.1) in numerical computations (in polar coordinates \((r, \theta)\) along the radial direction \( r \) of the region under consideration; an ordinary quadrature rule is used along the polar direction \( \theta \) with \( 0 \leq \theta \leq 2\pi \)). This remark was made in Reference 10. It was also reported in Reference 11 together with additional related comments.

We can add in passing that Rösel suggested the use of the term hyperintegrals for finite-part (or, equivalently, Hadamard-type) integrals in 1978 and, some years later, the term hypersingular integrals was suggested, since this term corresponds better to the related term singular integrals so frequently already used in applied mechanics and engineering. The latter term has been widely adopted by the engineering community and is that really mostly used in the literature. (Similarly, in the same report, Rösel suggested also the use of the analogous term hyperderivatives for fractional derivatives not of interest here.)

The use of finite-part integrals (after their mathematical introduction by Hadamard at the beginning of this century) in applied mechanics and engineering is really very old. The classical related field is fluid dynamics, where the results of Multhopp have been really pioneering and still constitute the classical related reference. On the contrary, in solid mechanics (plane and three-dimensional elasticity), it seems that finite-part (Hadamard-type and, in a special case, Mangler-type) integrals were used much later as a more or less natural way of formulating and solving the related boundary value problems always with emphasis to fracture mechanics. (Of course, it is understood that any boundary of a medium, external and internal boundaries, can be considered as a 'crack' with a second 'fictitious' medium having the same boundary; this is a very well known trick in the theory of elasticity with many important applications in the related numerical methods.)

In a rather long series of papers (not referenced here for the sake of brevity, but easily accessible in the scientific literature), several aspects of finite-part integrals were considered including the formulation of the hypersingular integral equations for the fundamental crack problems both in plane and in three-dimensional elasticity by a variety of direct and indirect methods (including Betti's reciprocal work theorem), the closed-form evaluation of these integrals in simple cases, the numerical computation of finite-part integrals by using appropriate quadrature rules, the numerical solution of the corresponding hypersingular integral equations, the principal value definition of these integrals, the Sokhotski–Plemelj formulae for the corresponding boundary values of the related analytic functions, an extension of the definition of finite-part integrals ideal for cracks along a bimaterial interface, etc.

Since the theory of elasticity is a really active part of recent research in numerical methods in engineering, several authors adopted Hadamard's concept of finite-part integrals in their research in elasticity problems (including, of course, fracture mechanics) and successfully reduced quite complicated boundary value problems in elasticity to hypersingular integral equations and/or investigated these equations either theoretically or from the practical/numerical-solution point of view. We can mention, e.g., the most interesting contributions by Lin'kov and Mogilevskaya, Rong, Martin and Rizzo and Mayrhofer and Fischer.

Today the method of finite-part integrals has reached a really very high power and become an essential part of numerical methods in elasticity problems (including fracture mechanics) and
engineering in general and the number of related research results (both in journals and in conference proceedings) increases exponentially. Among a very large number of such recent papers, we can make reference to the most interesting results by Sladek, Sladek and Tanaka, Gray and San Soucie, Lee, Advani and Lee, Ang and Noone and Qin and Tang. There are also several more very important papers making sufficient reference to the aforementioned results of Kutt on numerical integration rules for finite-part integrals and the related method in elasticity in general (see, e.g., Reference 24).

Therefore, it is clear that the use of finite-part integrals in acoustics, fluid dynamics, elasticity, fracture mechanics, etc. has already become a standard related approach from the engineering point of view with a very large number of related papers (mostly in journals and conferences devoted to numerical and computational methods in engineering). But, to the best of this author's personal knowledge, in spite of the fundamental results of Kutt and related results for the Gaussian quadrature rule for the fundamental finite-part integral \( I \) in (1.1) (the 'prototype' in these quadrature rules), the convergence of this rule has not been proved up to now. Because of the interest of this rule in numerical methods in engineering and its frequent real use in these methods (mainly for the computation of Cauchy-type principal value integrals in two dimensions as was suggested in Reference 10), we feel that this lack of proof is an essential disadvantage of this important quadrature rule and we will proceed to this proof below. (On the contrary, in almost all cases, the numerical integration rules used in engineering applications have their convergence results well established in numerical-analysis papers and books such as the classical monograph by Davis and Rabinowitz and, in few cases, the convergence results accompany the manuscript; see, e.g., the recent paper by Huang and Cruse.)

On the other hand, we cannot neglect the computational-mathematics-numerical-analysis point of view, where the nonclassical weight function \( 1/y \) on the interval \([0,1]\) in (1.1) is, in some way, a real challenge to researchers in numerical integration. As is clear from the tables of nodes and weights appeared in References 1 and 9, one (the first) node in the Gauss quadrature rule for \( I \) is negative therefore lying outside the integration interval \([0,1]\). Similarly, the corresponding weight is also negative. Clearly, this fact (the nonclassical form) of the present weight function \( 1/y \) (with a strong singularity at \( y = 0 \)) does not constitute a contradiction to the classical theory of construction of Gaussian quadrature rules and (see also the important related results by Tsamasphyros and Dimou), but, unfortunately, it does not permit the application of the related well-known convergence theory.

We can also mention at this point that more or less similar nonclassical (that is with the possibility of appearance of one or two nodes outside the integration interval and, further, with the related probable appearance of negative or even complex weights) Gaussian quadrature rules for particular (but very important in engineering applications) classes of Cauchy-type principal-value and finite-part integrals were suggested for the first time by Tsamasphyros and Dimou. Similarly, recently, the integration interval in (1.1) was generalized from \([0,1]\) to \([0,\infty)\), that is the Gauss–Laguerre quadrature rule for the finite-part integral \( I \) was constructed. All these three Gaussian quadrature rules for finite-part integrals are nonclassical Gaussian quadrature rules (with the related well-known theory of convergence failing to be directly applicable). Therefore, we feel it an interesting (both from the theoretical and from the practical point of view) possibility to prove the convergence of the Gauss (better the Gauss–Legendre) quadrature rule for \( I \) in (1.1) expecting that analogous results may appear in future for the quadrature rules in References 25 to 27 and additional nonclassical Gaussian quadrature rules. Of course, further approaches (beyond nonclassical Gaussian quadrature rules) are also available in the literature as well as efficient methods for the numerical solution of singular integral equations with finite-part integrals (hypersingular...
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In Reference 11, several theoretical results concerning the Gauss quadrature rule for \( I \) were proved. Among these results we can mention here: (i) That the orthogonal polynomials \( Q_n(y) \) \((n \geq 2)\) for the weight function \( 1/y \) on \([0,1]\) (in the finite-part interpretation of the related integrals) are simply of the form

\[
Q_n(y) = P_n^*(y) + a_nP_{n-1}^*(y), \quad n \geq 2,
\]

where \( P_n^*(y) \) denote the classical shifted Legendre polynomials, related to the Legendre polynomials (on \([-1,1]\)) \(^8,31\) by

\[
P_n^*(y) = P_n(x), \quad x = 2y - 1 \iff y = (x + 1)/2,
\]

and \( a_n \) is the constant \(^11\)

\[
a_n = \sum_{k=1}^{n} \frac{1}{k} \left/ \sum_{k=1}^{n-1} \frac{1}{k} \right. = 1 + \left[ n \sum_{k=1}^{n-1} \frac{1}{k} \right]^{-1} > 1.
\]

(This simple remark, relating the nonclassical polynomials \( Q_n(y) \) to the completely classical Legendre polynomials, \(^8,31\) seems having escaped the attention of Kutt and of the users of his results; in fact, Kutt constructed \( Q_n(y) \) from first principles.) (ii) The theoretical proof of the fact that the first node \( y_{1n} \) (and only this node) in the Gauss quadrature rule for \( I \) \(^1,2\)

\[
I = \sum_{i=1}^{n} A_{in} f(y_{in}) + R_n
\]

\((y_{in} \text{ denoting the nodes, } A_{in} \text{ the corresponding weights and } R_n \text{ the error term})\) is negative. Alternative, but equivalent, expressions for \( a_n (n \geq 2) \) and formulae for the weights \( A_{in} \) in (1.6) were also derived in Reference 11.

After this rather long, but also indispensable in our opinion (because of several misunderstandings and an incomplete literature on quadrature rules for finite-part integrals), introduction, we proceed now to the results of the present paper, the most important of which is the proof of the convergence of (1.6) in Section 5, which seems not proved by Kutt and other interested researchers up to now. Following our engineering education and experience, we would like to begin with some simple numerical results verifying this convergence for some well-behaved functions \( f(y) \).

2. NUMERICAL EXPERIMENTS

We applied the Gauss quadrature rule (1.6) for the numerical evaluation of \( I \) in (1.1) to several functions \( f(y) \) and with \( n = 2, 3, \ldots, 8 \) nodes. \(^1,2\) The obtained numerical results are displayed in the first part of Table I and they show the convergence of Kutt's Gauss quadrature rule for \( I \) (which will be proved theoretically in Section 5). Similarly, we applied the Radau quadrature rule (with the preassigned node \( y_{1n} = 0 \)) to the same finite-part integrals, again with \( n = 2, 3, \ldots, 8 \) nodes. \(^2\) This rule is completely equivalent to the classical Gauss–Legendre quadrature rule, \(^5–8\) but for the ordinary integral (1.2) (which is essentially the definition of (1.1)) on the interval \([0,1]\) \(^5\) with \( n - 1 \) nodes. \(^2\) The derived numerical results are presented in the second part of Table I.

From the results of Table I we conclude directly that the Gauss quadrature rule (of polynomial accuracy \( 2n - 1 \)) is always superior to the Radau quadrature rule (of polynomial accuracy \( 2n - 2 \))
Table I. Numerical results for the finite-part integral $I$ in (1.1) by using both the Gauss and the Radau (with the preassigned node $y = 0$) quadrature rules (of the form (1.6)) with $n = 2, 3, \ldots, 8$ nodes and for several integrands $f(y)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(y) = y^4$</th>
<th>$f(y) = e^y$</th>
<th>$f(y) = e^{-y}$</th>
<th>$f(y) = e^{-y} \cos y$</th>
<th>$f(y) = 1/(1 + y^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gauss quadrature rule</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.20833333</td>
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<tr>
<td></td>
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</tr>
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</table>

as is expected. Moreover, both rules converge, rather rapidly. We can add that the exact values of $I$ are: $I = 1/4$ for $f(y) = y^4$ (in this case, $I$ reduces, evidently, to an ordinary integral); $I = Ei(1) - \gamma = 1.317902151$ (where $Ei(y)$ is an exponential integral$^5$ and $\gamma$ Euler's constant) for $f(y) = e^y$ (as can easily be proved after elementary computations based on the Maclaurin series for $e^y$,$^{32}$ $I = -Ei(1) - \gamma = -0.7965995989$ (where $Ei(y)$ is another classical exponential integral$^5$) for $f(y) = e^{-y}$ (as can similarly be proved).

Of course, since the aforementioned Radau quadrature rule for (1.1) is identical to the Gauss–Legendre quadrature rule for (1.2), its convergence is assured for integrands $f(y)$ in (1.1) with a continuous first derivative (this restriction due to the form of the integrand in (1.2)) as is very well known.$^6$ Therefore, we will pay in the sequel no attention to the Radau quadrature rule, restricting ourselves to the more accurate and much less classical Gauss quadrature rule for the finite-part integral (1.1), where one negative node, $y_{1n}$, is always present. This rule is valid for $n \geq 2$.$^2,11$ The values of this negative node, $y_{1n}$, are displayed in the second column of Table II for $n = 2, 3, \ldots, 20$, obtained directly from Reference 1 for these values of $n$ (as well as from Reference 9 for $n = 2, 3, \ldots, 8$). From the numerical results of Table II we observe that the absolute value of $y_{1n}$ reduces to zero as $n \to \infty$, that is, this negative node approaches zero as $n \to \infty$. This fact will be investigated theoretically in the next section, where bounds for $y_{1n}$ will be established. These bounds will be used (together with additional results) in Section 5 for the establishment of
Table II. Comparison of the exact values of the negative node $y_{1n}$ of the Gauss quadrature rule for the finite-part integral (1.1) with their bounds determined from (3.25), $b_{1n}$, as well as from (3.1) or (3.29), $b_{2n}$, for $n = 2, 3, \ldots, 20$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$y_{1n}$</th>
<th>$b_{1n}$</th>
<th>$b_{2n}$</th>
<th>$a_n$</th>
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the convergence of the Gauss quadrature rule for the finite-part integral (1.1).

In passing, we notice that since $\int_0^1 (dy/y) = 0$, the sum of the weights $A_{in}$ is equal to zero in our case and, moreover, the weight $A_{1n}$ corresponding to $y_{1n}$ is negative and increasing in absolute value for increasing values of $n$.

3. BOUNDS FOR THE NEGATIVE NODE

It was proved in Reference 11 that the roots $y_{in}$ of $Q_n(y)$ (the orthogonal polynomial of degree $n$ of the set of orthogonal polynomials with respect to the weight function $1/y$ on $[0,1]$, determined by (1.3)) are real and distinct. Moreover, $n - 1$ of these roots, nodes of the Gauss quadrature rule (1.6), lie inside the integration interval $(0,1)$ of (1.1) and (1.2), whereas one of these nodes, $y_{1n}$, is negative, lying outside this interval. Here we will prove that:
Theorem 1

The absolute value of the negative node $y_{1n}$ of the Gauss quadrature rule for the finite-part integral (1.1) is bounded by $|b_{2n}|$, where

$$b_{2n} = -\left[ n^2(2\log n - 1) + n \right]^{-1},$$

(3.1)

or, equivalently, $b_{2n} < y_{1n} < 0$ for every value of $n$ ($n \geq 2$).

Proof. We are studying the smallest root $y_{1n}$ of $Q_n(y)$ ($n \geq 2$), defined by (1.3), in terms of two consecutive shifted Legendre polynomials. We already know from Reference 11 (as is also verified from the numerical results of Reference 2, displayed in the second column of Table II as well) that $y_{1n} < 0$. We find it convenient to use the new variable

$$t = 1 - 2y \quad \leftrightarrow \quad y = (1-t)/2, \quad \text{whence} \quad t_{1n} = 1 - 2y_{1n} > 1 \quad \text{or} \quad y_{1n} = (1-t_{1n})/2 < 0.$$  

(3.2)

Now, because of (1.4), $x = -t$ and we find from (1.3) and (1.4) that

$$Q_n(y) \equiv \tilde{Q}_n(t) = P_n(-t) + a_n P_{n-1}(-t)$$  

(3.3)

or, since $^8$

$$P_n(-t) = (-1)^n P_n(t),$$

(3.4)

we obtain from (3.3)

$$Q_n(y) \equiv \tilde{Q}_n(t) = (-1)^n[P_n(t) - a_n P_{n-1}(t)].$$

(3.5)

We are interested in the value $t_{1n} = 1 - 2y_{1n} > 1$ of the greatest positive root of the polynomial

$$(-1)^n\tilde{Q}_n(t) \equiv \tilde{Q}^*_n(t) = P_n(t) - a_n P_{n-1}(t).$$

(3.6)

We take into account the Rodrigues formula for the Legendre polynomials, concluding from this formula that both $P_n(t)$ and its derivatives are positive for $t > 1$. On the other hand, for $t = 1$ we have$^3$

$$P_n(1) = 1, \quad P_n'(1) = n(n + 1)/2.$$  

(3.7)

Therefore, because of (1.5),

$$\tilde{Q}^*_n(1) = 1 - a_n = -\left[ n \sum_{k=1}^{n-1} \frac{1}{k} \right]^{-1} < 0.$$  

(3.8)

Moreover, on the basis of (3.6) and the second of (3.7),

$$\tilde{Q}^{**}_n(1) = \left[ n(n + 1)/2 - a_n [n(n - 1)/2] \right] = n [(a_n + 1) - n(a_n - 1)]/2.$$  

(3.9)

But we find from (1.5) that for $n \geq 2$ (as is here the case)

$$0 < n(a_n - 1) = \left[ \sum_{k=1}^{n-1} \frac{1}{k} \right]^{-1} \leq 1.$$  

(3.10)
Hence, obviously,

\[ \tilde{Q}_n^{**}(1) > 0 \]  

(as is clear from (3.9)). Taking into account (3.8) and (3.11), we will prove the following generalization of (3.11)

\[ \tilde{Q}_n^{*(m)}(1) > 0, \quad m \geq 1. \]  

(3.12)

To this end, we will use two elementary properties of the classical Legendre polynomials \( P_n(x) \).

At first, we use the fact that \( P_n(x) \) satisfy the classical Legendre differential equation

\[ (1 - x^2)P''_n(x) - 2xP'_n(x) + n(n + 1)P_n(x) = 0. \]  

(3.13)

(For \( x = 1 \) we obtain from this differential equation the second of (3.7).) By successive differentiations of this equation we can easily prove (by induction) that

\[ (1 - x^2)P^{(m+2)}_n(x) - 2(m + 1)xP^{(m+1)}_n(x) + [n(n + 1) - m(m + 1)]P^{(m)}_n(x) = 0. \]  

(3.14)

((3.13) is a special case of (3.14), resulting for \( m = 0 \).) For \( x = 1 \) we obtain from (3.14) (for \( m - 1 \) instead of \( m \))

\[ \frac{n(n + 1) - m(m - 1)}{2m} P^{(m-1)}_n(1), \quad 1 \leq m \leq n. \]  

(3.15)

(For \( m = 1 \) (3.15) reduces to the second of (3.7) because of the first of (3.7).) We will use this formula below.

In a similar way, we use the following elementary property of the Legendre polynomials \( P_n(x) \)

\[ xP'_{n-1}(x) = P'_n(x) - nP_{n-1}(x). \]  

(3.16)

By successive differentiations of this equation, we can easily prove (by induction) that

\[ xP^{(m)}_{n-1}(x) = P^{(m)}_n(x) - (n + m - 1)P^{(m-1)}_{n-1}(x), \quad m \geq 1. \]  

(3.17)

((3.16) is a special case of (3.17), resulting for \( m = 1 \).) For \( x = 1 \) we obtain from (3.17)

\[ P^{(m)}_n(1) = P^{(m)}_{n-1}(1) + (n + m - 1)P^{(m-1)}_{n-1}(1), \quad m \geq 1. \]  

(3.18)

We have already proved (3.11). We will prove (3.12) by induction, taking into consideration that it holds true for \( m = 1 \) (because of (3.11)). At first, because of (3.6), we have

\[ \tilde{Q}_n^{*(m)}(1) = P^{(m)}_n(1) - a_n P^{(m)}_{n-1}(1). \]  

(3.19)

Taking into account (3.18), we rewrite (3.19) as

\[ \tilde{Q}_n^{*(m)}(1) = (n + m - 1)P^{(m-1)}_{n-1}(1) - (a_n - 1)P^{(m)}_{n-1}(1). \]  

(3.20)

Next, taking into account (3.15) (for \( n - 1 \) instead of \( n \)) as well, we rewrite (3.20) as

\[ \tilde{Q}_n^{*(m)}(1) = \frac{2m(n + m - 1) - (a_n - 1)[n(n - 1) - m(m - 1)]}{2m} P^{(m-1)}_{n-1}(1). \]  

(3.21)
For \( m = 1 \) \((3.21)\) reduces to (because of the first of \((3.7)\))

\[
\tilde{Q}^*_{n}^{(1)} = \frac{2n - (a_n - 1)n(n - 1)}{2} = n[(a_n + 1) - n(a_n - 1)]/2,
\]

which coincides with \((3.9)\). Therefore, \((3.12)\) holds true for \( m = 1 \) (because of \((3.11)\)). Thus, having proved \((3.12)\) for \( m = 1 \), we assume it valid for \( m > 1 \). Then it is clear from \((3.21)\) that this is the case since \( P_{n-1}^{(m-1)}(1) > 0 \) (because of \((3.15)\)) and the numerator in the fraction of \((3.21)\), which is positive for \( m = 1 \) (because of \((3.9)\) and \((3.22)\)), is an increasing function of \( m \) (since \( a_n > \pm 1 \) as is clear from \((1.5)\)). This completes the proof of \((3.12)\). Therefore, \( \tilde{Q}^*_{n}(t) \), although negative for \( t = 1 \) (because of \((3.8)\)), has all its derivatives positive at this point. Hence, it has only one root, \( t_{1n} \), for \( t > 1 \) (as was also observed in Reference 1, proved in Reference 11 and really expected).

Moreover, on the basis of the previous results, we conclude also directly that this sought root, \( t_{1n} \), of \( \tilde{Q}^*_{n}(t) \) is bounded by

\[
1 < t_{1n} < 1 + |\tilde{Q}^*_{n}(1)|/\tilde{Q}^*_{n}^{(1)}.
\]

Because of \((3.8)\) and \((3.9)\), we find that

\[
1 < t_{1n} < 1 + 2(a_n - 1)/\{n[(a_n + 1) - n(a_n - 1)]\}.
\]

Due to the last of \((3.2)\), \((3.24)\) can also be written as

\[
b_{1n} = -(a_n - 1)/\{n[(n + 1) - n(a_n - 1)]\} < y_{1n} < 0.
\]

This is the sought bound \( b_{1n} \) for \( y_{1n} \) in terms of \( a_n \), defined by \((1.5)\) and displayed in the last column of Table II for \( n = 2, 3, \ldots, 20 \). The numerical values of \( b_{1n} \) are presented (for \( n = 2, 3, \ldots, 20 \)) in the third column of Table II.

As is clear from these numerical values, displayed in Table II (where \( y_{1n} \) and \( b_{1n} \) can be compared), the bound \((3.25)\) for \( y_{1n} \) is sufficiently sharp and satisfactory. This means that the difference between \( b_{1n} \) and \( y_{1n} \) is not large (especially for increasing values of \( n \)) although, of course, we have always \( b_{1n} < y_{1n} < 0 \). Yet, since this bound contains the constant \( a_n \), determined from \((1.5)\), we cannot use it very easily. A less sharp, but more convenient in practice, bound, \( b_{2n} \), given by \((3.1)\), can be found as follows. By taking into account the method of comparing sums with integrals,\(^3\) we obtain directly the classical result

\[
\sum_{k=1}^{n-1} \frac{1}{k} \geq \log n, \quad n \geq 2.
\]

Then \((1.5)\) yields

\[
a_n \leq 1 + (n \log n)^{-1}
\]

and \((3.25)\) reduces to the following less strict form

\[
b_{2n} \equiv -\{n^2 \log n\{(n + 1) - [1 + (n \log n)^{-1}](n - 1)\}\}^{-1} < y_{1n} < 0
\]

and, finally, to

\[
b_{2n} \equiv -\{n(2n \log n - n + 1)\}^{-1} < y_{1n} < 0,
\]
A QUADRATURE RULE FOR FINITE-PART INTEGRALS

which is completely equivalent to (3.1). This completes the proof of the theorem. □

The numerical values of \( b_{2n} \) are also displayed (for \( n = 2, 3, \ldots, 20 \)) in the fourth column of Table II. (For convenience, we display also the values of \( a_n \), again for \( n = 2, 3, \ldots, 20 \) in the last column of Table II.)

From the results of this section we conclude that for sufficiently large values of \( n \) all the nodes \( y_{in} \) of the Gauss quadrature rule (1.6) for the finite-part integral (1.1) lie in an interval \((-\delta, 1)\), where \( \delta \) is an arbitrary small positive quantity. For example, if \( \delta \) is selected equal to \(-y_{12} \approx 0.1319\), then all the nodes \( y_{in} \) for all values of \( n \) (\( n \geq 2 \)) lie in \((-\delta, 1)\). This result is of fundamental importance for the convergence results of Section 5. Moreover, from Theorem 1 it is concluded that for \( n \to \infty \), \( y_{in} \to 0^- \). This is also clear from the numerical results of Table II. Before proceeding to the proof of the convergence of the aforementioned quadrature rule, we will consider the sign of the weights \( A_{in} \) since we will need it during this proof.

4. SIGN OF THE WEIGHTS
At first, we observe from the numerical results for the nodes and weights\(^1,^9\) that all the weights \( A_{in} \) \((i = 2, 3, \ldots, n)\) corresponding to positive nodes are always positive, whereas the weight \( A_{1n} \), corresponding to the negative node \( y_{1n} \), is always negative. We found it convenient to introduce the new, auxiliary weights \( B_{in} \) defined by

\[
B_{in} = A_{in} y_{in}, \quad i = 1, 2, \ldots, n. \tag{4.1}
\]

Then the fundamental quadrature rule (1.6) takes the more natural form

\[
I = \sum_{i=1}^{n} B_{in} \frac{f(y_{in})}{y_{in}} + R_n. \tag{4.2}
\]

Clearly, as is concluded from the aforementioned numerical values for \( y_{in} \) and \( A_{in} \), all the auxiliary weights \( B_{in} \) \((i = 1, 2, \ldots, n)\) are now positive. (In passing, we can mention that the quadrature rule (4.2) is not very strange if (1.1) is taken into account.) We will prove the above remarks on the sign of the original weights \( A_{in} \) and the auxiliary weights \( B_{in} \) \((i = 1, 2, \ldots, n)\). Of course, we notice that since \( 1/y \) is not a classical weight function, \( Q_n(y) \) do not present the well-known properties of orthogonal polynomials, like some elementary properties (e.g. the Christoffel–Darboux formula in its classical form\(^6,^8\)) on which the positive sign of the weights \( A_{in} \) may be based. (Moreover, \( Q_1(y) \) does not exist at all.) This fact permits the weights \( A_{in} \) to be of both signs (in our case, \( A_{1n} \) is negative, whereas \( A_{in}, i = 2, 3, \ldots, n, \) are positive). In spite of these difficulties, it has become possible to prove the aforementioned properties for the sign of the original weights \( A_{in} \) and the auxiliary weights \( B_{in} \). We begin with \( B_{in} \). We will prove the following theorem:

**Theorem 2**

All the auxiliary weights \( B_{in} \) \((i = 1, 2, \ldots, n)\), defined by (4.1), are positive.

**Proof.** We will use (4.2) together with (1.1) for the definition of \( I \). Since we are considering the Gauss quadrature rule for \( I \), the error term \( R_n \) vanishes for \( f(y) \) being a polynomial of degree up to \( 2n - 1 \).\(^2\) Therefore, appropriately modifying a classical device for the proof of the positiveness
of the weights in classical Gauss quadrature rules\(^8\) so that we can get rid (even in our case) of the finite-part interpretation of \(I\) in (1.1), we apply (4.2) for \(f(y) \equiv f_{kn}(y)\), where

\[
f_{kn}(y) = y l_{kn}^2(y), \quad k = 1, 2, \ldots, n,\]

(4.3)

where \(l_{kn}(y)\) are the fundamental polynomials associated with the Langrange classical interpolation formula (based on the nodes \(y_i, i = 1, 2, \ldots, n\)) in our case, that is,\(^6-8,33\)

\[
l_{kn}(y) = \frac{Q_n(y)}{Q_n'(y_{kn})(y - y_{kn})}, \quad k = 1, 2, \ldots, n, \quad n \geq 2,\]

(4.4)

whence (with \(\delta_{ik}\) Kronecker’s delta)

\[
l_{kn}(y_{kn}) = \delta_{ik}, \quad i, k = 1, 2, \ldots, n.\]

(4.5)

Therefore, since \(l_{kn}(y)\) is a polynomial of degree \(n - 1\), \(f(y)\) in (4.3) is really a polynomial of degree \(2n - 1\) and, in this way, \(R_n = 0\). Because of (4.5), we obtain in our case from (1.1) and (4.2)

\[
I = \int_0^1 \frac{f_{kn}(y)}{y} \, dy = \int_0^1 l_{kn}^2(y) \, dy = B_{kn}, \quad k = 1, 2, \ldots, n,\]

(4.6)

which is clearly a positive quantity (since \(I\) is now an ordinary integral) even for \(B_{1n}\), corresponding to the node \(y_{1n}\) outside \([0,1]\). \(\square\)

As a corollary of the previous theorem, we can also prove the following fundamental result concerning now the original weights \(A_{in}\) in (1.6):

**Theorem 3**

All the original weights \(A_{in}\) corresponding to a positive node \(y_{in}\) \((i = 2, 3, \ldots, n)\) are positive; the original weight \(A_{1n}\), corresponding to the negative node \(y_{1n}\), is negative.

**Proof.** It is sufficient just to take into consideration Theorem 2 and the definition (4.1) of the auxiliary weights \(B_{in}\) \((i = 1, 2, \ldots, n)\). \(\square\)

This theorem completely explains the signs of the weights \(A_{in}\) tabulated (in our case) by Kutt\(^1\) (see also Reference 9). Finally, we observe that since the integrand in (4.6) is a polynomial of degree \(2n - 2\) for an ordinary integral, we can evaluate \(I\) in (4.6) (with no error term) by using the classical Gauss–Legendre (or one of the two Radau–Legendre) quadrature rule for the integration interval \([0,1]\). (We have used this property in order to verify the validity of (4.6) numerically.) We can proceed now to the fundamental result of this paper: the proof of the convergence of (1.6) or, in the notation of this section, equivalently, of (4.2). In fact, in practice, (4.2) is more convenient than (1.6) since both it is more natural in appearance and, what is more important, all of the weights \(B_{in}\) in (4.2) (having been called auxiliary weights) are positive numbers (because of Theorem 2).

5. CONVERGENCE RESULTS

We will prove the convergence of the Gauss quadrature rule (1.6) or (4.2) for (1.1) by modifying the classical proof of convergence of Gaussian quadrature rules for continuous functions\(^6,7\) so that it can become applicable to our case. An alternative possibility would be to attempt to appropriately
modify the convergence proofs for Gaussian quadrature rules for Cauchy-type principal value integrals so that they can also become applicable to finite-part integrals. Such a proof, for Cauchy-type integrals, was suggested by Criscuolo and Mastroianni.\(^\text{34}\) This is essentially the approach having been followed in Reference 35, but for a completely different class of finite-part integrals. Yet, here we preferred the direct approach of proof, essentially based on the equivalent rewriting of (1.1) in the form (1.2) and the resulting ‘transformation’ of the finite-part integral (1.1) into an ordinary integral, (1.2), simply since we found this approach more convenient and natural. More specifically, we will prove the following theorem, where we assume the continuity of the first derivative \(f'(y)\) of \(f(y)\) on \([-\delta, 1]\) (with \(\delta\) defined in the last paragraph of Section 3):

**Theorem 4**

Let \(f(y) \in C^1[-\delta, 1]\). Then

\[
\lim_{n \to \infty} R_n(f) = 0. \tag{5.1}
\]

**Proof.** We consider an interval \([-\delta, 1]\) including all the nodes \(y_{in}\) \((i = 1, 2, \ldots, n)\) for all values of \(n \geq 2\). This is possible if the results of Section 3 (especially Theorem 1) are taken into account. \((\delta\) may be as small as at least 0.1319, Table II.) In order to become able to apply in the present case the aforementioned classical results for the convergence of the Gauss quadrature rule, we take into consideration the equivalent form (1.2) of the finite-part integral (1.1). In this form, we have simply an ordinary integral. We define also the new, auxiliary integrand

\[
\phi(y) = [f(y) - f(0)]/y. \tag{5.2}
\]

Then, by taking into account the elementary mean-value theorem for the derivative of a function\(^7\) and the assumption that \(f'(y)\) is continuous on \([-\delta, 1]\), we conclude that \(\phi(y)\) is also continuous on \([-\delta, 1]\). Then (1.2) takes the equivalent, but more elementary in appearance, form

\[
I = \int_0^1 \phi(y) \, dy. \tag{5.3}
\]

Moreover, (1.6) can also be written as

\[
I = \sum_{i=1}^n A_{in} f(y_{in}) + R_n(f). \tag{5.4}
\]

But since

\[
\int_0^1 \frac{dy}{y} = \sum_{i=1}^n A_{in} = 0 \tag{5.5}
\]

(as was already mentioned in the last paragraph of Section 2), (5.4) can further be written in the following completely equivalent form

\[
I = \sum_{i=1}^n A_{in} [f(y_{in}) - f(0)] + R_n(f). \tag{5.6}
\]
Because of (5.2) together with the definition (4.1) of the auxiliary weights $B_{in}$, we finally have the auxiliary quadrature rule

$$I = \sum_{i=1}^{n} B_{in} \phi(y_{in}) + R_n^*(\phi), \quad R_n^*(\phi) \equiv R_n(f),$$

(5.7)

with nodes $y_{in}$ in $(-\delta, 1)$ (because of Theorem 1), positive weights (because of Theorem 2) and a continuous integrand $\phi(y)$ (because of the related assumption on $f(y)$ in the present theorem and the above results). We are now ready to proceed to the required slight modification of the proof of the convergence for the classical Gauss quadrature rules.\textsuperscript{6,7}

In this way, by using the classical Weierstrass approximation theorem in approximation theory,\textsuperscript{33} we can find for a given $\varepsilon > 0$ a polynomial $p_m(y)$ of a sufficiently large degree $m$ such that

$$\max_{-\delta \leq y \leq 1} |\phi(y) - p_m(y)| < \varepsilon.$$  

(5.8)

Moreover, we know that

$$R_n^*(\phi) = R_n^*(p_m) + R_n^*(\phi - p_m).$$  

(5.9)

For all $n$ such that $m \leq 2n - 2$, $R_n^*(p_m)$ obviously vanishes and (5.9) reduces to

$$R_n^*(\phi) = R_n^*(\phi - p_m).$$  

(5.10)

Finally, by taking into account both (5.3) and (5.7), we find that

$$|R_n(f)| = |R_n^*(\phi)| = |R_n^*(\phi - p_m)|$$

$$= \left| \int_{-1}^{1} [\phi(y) - p_m(y)] \, dy - \sum_{i=1}^{n} B_{in} [\phi(y_{in}) - p_m(y_{in})] \right|$$

$$\leq \int_{-1}^{1} |\phi(y) - p_m(y)| \, dy + \sum_{i=1}^{n} B_{in} |\phi(y_{in}) - p_m(y_{in})| \leq 2\varepsilon$$

(5.11)

because of (5.8), the fact that $B_{in}$ are all positive, as well as the elementary formula

$$\int_{0}^{1} dy = \int_{0}^{1} y^{-1} y \, dy = \sum_{i=1}^{n} A_{in} y = \sum_{i=1}^{n} B_{in} = 1,$$  

(5.12)

resulting directly from the definition (4.1) of $B_{in}$. (5.11) completes the proof of the theorem. \textsuperscript{\square}

From the above theorem it is obvious that we can also find rates of convergence if $f(y)$ possesses higher-order derivatives on $[-\delta, 1]$. Thus, assuming that $f(y)$ has a continuous $k$th derivative on $[-\delta, 1]$, we can prove the following theorem for the rate of convergence of the present quadrature rule:

**Theorem 5**

Let $f(y) \in C^k[-\delta, 1]$ ($k > 1$). Then, if $2n - 2 > k$,

$$|R_n(f)| \leq An^{-k+1},$$

(5.13)
where $A$ is an appropriate positive constant depending on $f(y)$, but independent of $n$.

**Proof.** We take into account the classical elementary results of approximation theory (generalizations of Jackson's theorem\textsuperscript{33}). Then, since $f(y) \in C^k[\delta, 1]$ (by assumption), $\phi(y) \in C^{k-1}[\delta, 1]$ and, further, for a sufficiently large $m (m > k)$ there exists a polynomial $p_m(y)$ such that

$$\max_{-\delta \leq y \leq 1} |\phi(y) - p_m(y)| \leq Bm^{-k+1}. \quad (5.14)$$

In this relation, $B$ is an appropriate positive constant depending both on $f(y)$ (or, equivalently, $\phi(y)$) and on $k$, but not on $m$. Moreover, the facts that $\phi(y) \in C^{k-1}[\delta, 1]$, whence $\phi^{(k-1)}(y) \in C[\delta, 1]$, and, therefore, $|\phi^{(k-1)}(y)| \leq M$ on $[-\delta, 1]$, with $M$ a positive constant, were also taken into consideration.

Then it is clear from (5.11), because of (5.14), that

$$|R_n(f)| \leq 2Bm^{-k+1} \quad (5.15)$$

with $m \leq 2n - 2$. Of course, $n = (m + 3)/2$ is the best selection of $n$ in our case although the rate of convergence in (5.13), resulting directly from (5.15), does not improve in this case. (Only the value of $A$ in (5.13) becomes smaller if $m = 2n - 2$.)

### 6. DISCUSSION—GENERALIZATIONS

In the previous section, we have presented a proof for the convergence of the quadrature rule under consideration and established an elementary theorem on the rate of convergence. We have not been exhaustive neither has this been our intention. We omitted even to mention that we require $f'(y)$ to be continuous only in a neighbourhood of $y = 0$ in Theorem 4. The continuity of $f(y)$ is sufficient in the remaining part of $[-\delta, 1]$ as is clear from (5.2). (An analogous argumentation holds true for Theorem 5 with the $k$th derivative of $f(y)$ used instead of the first one.) On the other hand, the case of Hölder-(or Lipschitz-)continuous functions on $[-\delta, 1]$ or functions with a Hölder-continuous $k$th derivative\textsuperscript{33} is another important point of possible generalization of Theorem 5. We will not enter into the related details.

What has been the main purpose of this paper was to show that the classical elementary convergence results for the ordinary Gauss quadrature rules\textsuperscript{6,7} can more or less easily be modified and appropriately generalized to become applicable to the case of finite-part integrals under consideration (with the generalized weight function $1/y$). This has been achieved by using a slightly extended 'auxiliary' interval $[-\delta, 1]$ instead of the real integration interval $[0, 1]$, where $\delta$ can become as small as is required for sufficiently large values of $n$, the auxiliary (and always positive) weights $B_{in}$, defined by (4.1), instead of the original weights $A_{in}$, and the auxiliary integrand $\phi(y)$, defined by (5.2), instead of the original integrand $f(y)$.

Of course, the above results permit us to use the same ideas for other classes of generalized weight functions, e.g. $y^{-p}$ ($p > 1$), where pairs of complex conjugate nodes and weights may appear as well. ($p$ need not be restricted to integer values.) Nevertheless, this particular generalization seems not to be trivial. It is believed that with appropriate generalizations of the present results to other classes of finite-part integrals (with variable-signed or even pairs of complex conjugate nodes and weights), it will become possible to have theoretically established results for the location of the nodes, the signs both of the nodes and of the weights and, finally, the convergence and the rate of convergence of the related Gauss quadrature rules. This will permit the wide use of
this class of integrals (finite-part integrals) without the presently existing scepticism. This is particularly applicable to the quadrature rules for Cauchy-type and finite-part integrals suggested by Tsamasphyros and Dimou.25,26 (The Gauss–Laguerre rule for finite-part integrals, already having been proposed in Reference 27, can also be similarly studied.)

The reference (in 1984) by Davis and Rabinowitz6 of the Gauss quadrature rule for finite-part integrals (due to Kutt1,2) (in a classical monograph on numerical integration) has been the first step towards the official introduction of this class of quadrature rules in classical numerical integration theory. (It seems really unfortunate that finite-part integrals were ignored in a recent very interesting and practical book on numerical integration.36) We believe that the present paper will have also a small contribution to the same direction and that it will be followed by additional and, most probably, more important and useful related results.

The continuous appearance of research papers on hypersingular integrals/integral equations and their engineering applications in the computational mathematics37 and the engineering38–41 literature seems to be really very welcome. Nevertheless, unfortunately, the pure numerical-analysis literature on the solution of singular integral equations seems still mostly restricted (in spite of the appearance of the results of Golberg30) to Cauchy-type singular integral equations42 (with a really excellent extensive review also by Golberg43). This situation is, of course, slightly compensated by the fact that finite-part integrals were recently considered by Zwillinger in his integration handbook44 (with appropriate reference to some of the aforementioned related results) and, what is equally important, their numerical evaluation (to which this paper has been devoted) found a possible alternative (at least in special cases): the symbolic evaluation of finite-part integrals by using computer-algebra software. The paper by Gray45 is a short, but interesting, contribution towards this direction (concerning finite-part integrals appearing during the solution of the classical Laplace equation in two dimensions by using hypersingular integral equations) with the help of the well-known computer algebra system Maple,46 which has been already repeatedly used during the last two years in several applied-mechanics and engineering applications.

Finally, the question about the interest in quadrature rules for finite-part integrals and, especially, those under consideration in the present paper is very simple and has been already partially replied in Section 1: this class of integrals appears quite frequently in many branches of engineering. As far as the present (we dare say already classical) Kutt’s Gaussian quadrature rule is concerned, Telles47 has tested it during the computation of two-dimensional Cauchy-type principal value integrals (using polar coordinates as was already mentioned above) and found the whole approach ‘to produce very accurate results with a small number of integration points’. On the contrary, Schwab and Wendland48 preferred the interpolatory Kutt’s quadrature rule for hypersingular integrals (during the computation of finite-part integrals in boundary element methods in three dimensions), therefore having got rid of the nonclassical node outside the integration interval. Yet, in our opinion, in all cases (including the present one), Gaussian quadrature rules are superior (as far as their accuracy is concerned) to non-Gaussian interpolatory quadrature rules.

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