Design of an Electronically Controlled Non-Integer Order Generalized Filter

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M.Sc. Thesis

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Dedicated to my family for everything,
professor Costas Psychalinos, and to Alex for supporting me.
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Abstract

This Thesis focuses on the design and implementation of non-integer order filters derived from a first-order mother transfer function. Specifically, Fractional-order, Power-Law, and Double-order Low-pass, High-pass, and Band-pass filters, as well as their Inverse counterparts, are investigated. To effectively realize these filters, the derived transfer functions require approximation. While conventional approximation methods suffice for simulating the behavior of simple fractional operators such as the Laplacian operator $s^\alpha$, they prove inadequate for more complex filters, that involve more than one non-integer order coefficients. Thus, a curve fitting method based on MATLAB’s `fitfrd` command is employed. The resulting integer-order transfer functions can be implemented using conventional filter design topologies. This study utilizes the IFLF (Inverse-Follow-the-Leader-Feedback) multiple-feedback filter topology with OTAs (Operational Transconductance Amplifiers) as the active element, leveraging the adjustability of the transconductance through fine-tuning the DC bias currents. This structure offers flexibility by enabling implementation of multiple filter transfer functions from the same active core, through bias current adjustments. Post-layout simulation results, obtained using Cadence IC design software and the Design Kit provided by the Austria Mikro Systeme (AMS) 0.35 $\mu$m CMOS technology, validate the reliability of this approach. In summary, this proposed concept offers advantages such as resistorless implementations, satisfactory accuracy, utilization of only grounded capacitors, and a universally adaptable electronic structure.

**Keywords:** Fractional calculus, Fractional-order circuits, Fractional-order filters, Power-Law filters, Double-order filters, Generalized filters, CMOS integrated circuits, Operational Transconductance Amplifier
Περίληψη

Η παρούσα μεταπτυχιακή εργασία επικεντρώνεται στη σχεδίαση και υλοποίηση φίλτρων μη ακέραιας τάξης που προέρχονται από μια μητρική συνάρτηση μεταφοράς πρώτης τάξης. Συγκεκριμένα, διερευνώνται φίλτρα κλασματικής τάξης, κλασματικής δύναμης και διπλής τάξης βαθυπερατών, υψηπερατών και ζωνοδιαβατών, καθώς και τα αντίστοιχα αντίστροφα φίλτρα. Για την αποτελεσματική υλοποίηση αυτών των φίλτρων, οι παραγόμενες συναρτήσεις μεταφοράς απαιτούν προσέγγιση. Ενώ οι συμβατικές μέθοδοι προσέγγισης αρκούν για την προσομοίωση της συμπεριφοράς απλών κλασματικών τελεστών, όπως ο Λαπλασιανός τελεστής $s^α$, αποδεικνύονται ανεπαρκείς για πιο σύνηθες φίλτρα, που περιλαμβάνουν περισσότερους από έναν συντελεστή μη ακέραιας τάξης. Έτσι, χρησιμοποιείται μια μέθοδος προσαρμογής καμπυλών που βασίζεται στην εντολή `fitfrd` του MATLAB. Οι προκύπτουσες συναρτήσεις μεταφοράς ακέραιας τάξης μπορούν να υλοποιηθούν με τη χρήση συμβατικών τοπολογιών σχεδιασμού φίλτρων. Η παρούσα μελέτη χρησιμοποιεί την τοπολογία φίλτρου πολλαπλής ανάδρασης IFLF (Inverse-Follow-the-Leader-Feedback) με OTAs (Τελεστικοί ενισχυτές διαγωγιμότητας) ως ενεργό στοιχείο, αξιοποιώντας τη δυνατότητα ρύθμισης της διαγωγιμότητας μέσω της ρύθμισης των DC ρεύματων πόλωσης. Αυτή η δομή προσφέρει ευελιξία, επιτρέποντας την υλοποίηση τοπολογιών υποκαταβολής μεταφοράς φίλτρου από τον ίδιο ενεργό πυρήνα, μέσω προσαρμογών του ρεύματος πόλωσης. Τα αποτελέσματα της προσομοίωσης, που προέκυψαν με τη χρήση του λογισμικού σχεδιασμού ολοκληρωμένων κυκλωμάτων της Cadence και το Design Kit που παρέχεται από την τεχνολογία Austria Mikro Systeme (AMS) CMOS 0.35 μm, επικυρώνουν την αξιοπιστία αυτής της προσέγγισης. Συνοπτικά, η προτεινόμενη ιδέα προσφέρει πλεονεκτήματα όπως είναι η υλοποίηση χωρίς αντιστάσεις, ικανοποιητική ακρίβεια, χρήση μόνο γειωμένων πυκνωτών και μια καθολικά προσαρμοσμένη ηλεκτρονική δομή.

Λέξεις κλειδιά: Κλασματικός λογισμός, Κυκλώματα κλασματικής τάξης, Φίλτρα κλασματικής τάξης, Φίλτρα κλασματικής δύναμης, Φίλτρα διπλής τάξης, Γενικευμένα Φίλτρα, CMOS ολοκληρωμένα κυκλώματα, Ενισχυτής διαγωγιμότητας

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List of Abbreviations and Symbols

Abbreviations

ADE  Analog Design Environment
AMS  Austria Mikro Systeme
BP   Band-pass
BP,FO Fractional-order Band-pass
BP,GEN Generalized-order Band-pass
BP,PL Power Law Band-pass
CFE  Continued Fraction Expansion
CFOA Current-Feedback Operational Amplifier
CMOS Complementary Metal-Oxide-Semiconductor
DO   Double-order
FBD  Functional Block Diagram
FLF  Follow-the-Leader Feedback
FO   Fractional-order
FOMCON Fractional-order Modeling and Control
FPAA Field-Programmable Analog Array
HP   High-pass
HP,FO Fractional-order High-pass
List of Abbreviations and Symbols

HP,GEN Generalized-order High-pass
HP,PL Power Law High-pass
IBP Inverse Band-pass
IBP,FO Fractional-order Inverse Band-pass
IBP,GEN Generalized-order Inverse Band-pass
IBP,PL Power Law Inverse Band-pass
IC Integrated Circuit
IFLF Inverse Follow-the-Leader Feedback
IHP Inverse High-pass
IHP,FO Fractional-order Inverse High-pass
IHP,GEN Generalized-order Inverse High-pass
IHP,PL Power Law Inverse High-pass
ILP Inverse Low-pass
ILP,FO Fractional-order Inverse Low-pass
ILP,GEN Generalized-order Inverse Low-pass
ILP,PL Power Law Inverse Low-pass
IO Integer-order
LP Low-pass
LP,FO Fractional-order Low-pass
LP,GEN Generalized-order Low-pass
LP,PL Power Law Low-pass
MFB Multiple-Feedback
MOS Metal-Oxide-Semiconductor
OTA Operational Transconductance Amplifier
List of Abbreviations and Symbols

PL Power-law
VCCS Voltage-Controlled Current Conveyor

Symbols

$\omega_h$   Half-power frequency
$\omega_{peak}$ Peak frequency
$G$   Scaling factor
$G_{max}$ Maximum Gain
$G_{min}$ Minimum Gain
$g_m$ Transconductance
$I_B$ Bias current
$n_s$ Slope factor of a MOS transistor in the subthreshold region
$s$ Laplace operator
$V_{DD}$ Positive supply voltage
$V_{SS}$ Negative supply voltage
$V_T$ Thermal voltage
BW Bandwidth
Chapter 1

Introduction

1.1 Fractional calculus - Non integer-order filters

Fractional calculus, despite its origins in mathematical theory dating back over two centuries, has historically remained an arcane subject reserved for the mathematical elite. However, its reappearance in the twentieth century, combined with upgraded methodologies, has put it into the forefront of interdisciplinary research, particularly in biology, chemistry, and engineering [1]. This is due to its fundamental premise, which extends beyond conventional derivatives and integrals to include fractional orders unlike classical calculus, which deals exclusively with integer orders. This shift in perspective requires a re-evaluation of mathematical frameworks and analytical tools, paving the way for new approaches to problem-solving.

The definitions of fractional derivatives, namely the Riemann-Liouville definition, and the Grünwald-Letnikov approximation for discrete systems [2], are fundamental to current research in this area. These definitions, although challenging, accurately characterise fractional-order phenomena and enable investigation. The compatibility of fractional calculus with the Laplace transform is a major advance, that has enabled a seamless transition from time domain representations to frequency domain analysis. Non-integer order filters offer enable more precise control over the frequency response characteristics.

To realise non-integer order filters at circuit level [3], an intermediate step is necessary to approximate their transfer functions. Several approximation methods have been proposed in the literature to obtain the corresponding integer-order transfer function from a non-integer-order one [4, 5]. Such approximations are namely the Continued Fraction Expansion (CFE) [6], Oustaloup [7], Laguerre [8, 9], Matsuda [10, 11], Carlson and least-square methods for fitting the original frequency response. Some researchers have
addressed optimization problems, regarding parameter estimations for ultracapacitors or the Teaching learning-based optimization algorithm (TLBO) [12–14].

Fractional calculus provides a promising toolkit for optimising a wide range of real-world applications, particularly those following non-integer order physical principles [15]. Biomedical engineering applications [16], thermal systems and heat transfer phenomena [17], electrical noise [18], dielectric polarization [19], robotics [20], systems with long-range interactions [21], materials [22], all exemplify systems effectively described by such principles. Numerous studies have been conducted to comprehend the behaviour of such systems [23], in the field of improving control theory [24], concerning in particular renewable energy [25, 26], as well as the construction of fractional devices [27], capacitors and inductors of variable order, or even ultra-capacitors [13, 28], since the expression of their dynamic characteristics is superior through fractional calculus compared to classical approaches.

1.2 Thesis objectives

The main objective of this Thesis is to design and implement non-integer order filters derived from a first order filter transfer function. The concept of generalized filters is introduced and analysed, demonstrating that by adjusting the exponent values in accordance with the desired outcome, any category of 'order' and filter scenario becomes feasible. Specifically, the research includes the analysis and implementation of Low-pass, High-pass, and Band-pass filters, as well as their corresponding inverse counterparts. Moreover, a secondary aim is to effectively approximate the acquired transfer functions using curve fitting techniques within MATLAB. The Inverse-Follow-the-Leader-Feedback (IFLF) multiple-feedback filter topology is utilized to implement these approximated transfer functions, employing Operational Transconductance Amplifiers (OTAs) as active elements. By deploying the adjustability of the OTAs’ transconductance, this approach allows electronic tuning of the filter type, order and characteristic frequencies. Notably, this method offers the advantage of realizing all studied filters with a single core structure, facilitated by adjustment of the OTAs’ DC bias currents.

1.3 Thesis overview

This Thesis is organised in eight chapters, beginning with this introductory chapter and is organised as follows:
In Chapter 2, the mathematical foundation for comprehending non-integer order filters is presented. The chapter focuses on fractional order filters of order less than one and demonstrates how they can be derived from a first-order mother function. The transfer functions for Fractional order Low-pass, High-pass, and Band-pass filters are provided. Moreover, the process for obtaining the corresponding Inverse filters’ transfer functions is presented. The gain and phase frequency responses, as well as the cut-off frequencies, are then calculated for each case of filter.

Chapter 3 follows the same framework as Chapter 2 but focuses on the Power-Law filter category.

Chapter 4 introduces the concept of Generalized filters and illustrates how, by utilizing them and modifying the exponent values, the aforementioned two categories may be obtained. Likewise, the transfer functions are provided, alongside their frequency domain gain and phase responses, as well as the half-power frequencies. This chapter emphasizes the benefits of double-order filter transfer functions.

Chapter 5 explores the limitations of typical approximation approaches for addressing Generalized filters. Consequently, the curve fitting approximation is proposed as a feasible solution to this gap. The approach to obtaining an integer order approximate transfer function generated by a non-integer one is described, followed by a thorough examination of the corresponding MATLAB code. In addition, the design examples are presented, including their exponent values and the coefficients of the approximate transfer function, along with the associated gain responses graphs for each filter category computed using MATLAB.

Chapter 6 introduces the IFLF topology, employed for the implementation of the aforementioned transfer functions. Specifically, it provides the 4th order IFLF and its corresponding approximate transfer function, alongside the 4th-order OTA-C topology, serving as a versatile circuit for realizing all filters. Tables are included to detail the characteristics of each filter case, encompassing transconductances and finely-tuned bias currents for each OTA stage.

In Chapter 7, the performance evaluation of the derived structure is conducted through schematic and layout design using the Cadence IC design suite and MOS transistor models from the AMS 0.35 process. Additionally, post-layout simulations, Monte-Carlo analysis, Total Harmonic Distortion (THD) analysis, and dissipation characteristics are presented.

Finally, in Chapter 8, the conclusions reached within the context of this Thesis, as well as recommendations for future research, are presented.
Chapter 2

Fractional-Order Filters

2.1 Introduction

Fractional calculus is a branch of calculus that applies the principles of differentiation and integration to non-integer orders. This mathematical tool is a generalisation of integer calculus and is gaining interest in research due to its potential to simulate complex natural phenomena with greater accuracy than its conventional counterparts [29]. It is utilized in several fields of research, including engineering, control theory, materials theory, robotics, and biomedical signal processing.

Fractional-order filters are non-integer order filters that allow for more exact control over frequency characteristics by specifying the filter order as a fraction $\cdot$ such as 0.5 or $\pi$. The fractional-order is achieved by inserting one or more parameters of non-integer value, applied directly to the Laplace variable $s$, as follows: $\tau s \rightarrow (\tau s)^{\alpha}$, where $0 < \alpha < 1$. As a result, the derived function contains an additional independent parameter, which implies an additional degree of freedom. Adjusting the fractional-order $\alpha$ allows more precise control of important features of the filter, including the half-power frequency and the slope of the transition from the passband to the stopband.

This chapter describes the fractional-order transfer functions that can be obtained by switching from integer to fractional calculus in the case of different types of filters.
2.2 Fractional-order transfer functions

To generate all potential transfer functions for the conventional filter cases, a reference function is employed, referred to as the mother function hereinafter. Specifically, the mother function for a low-pass filter is as expressed in the equation:

\[
H_{LP}(s) = \frac{1}{\tau s + 1} = \frac{\omega_0}{s + \omega_0}
\]  

(2.1)

where \(\tau\) is a time constant related to the pole frequency as follows: \(\tau = 1/\omega_0\).

Below is the transfer function generated for a fractional-order low-pass filter by substituting the Laplace variable \(s\) with its fractional counterpart \(s^\alpha\), briefly denoted as \(LP,FO\):

\[
H_{LP,FO}(s) = G_0 \cdot \frac{1}{(\tau s)^\alpha + 1}
\]  

(2.2)

In the given expression, \(\alpha\) is the fractional order, with values between \(0 < \alpha < 1\) and \(G_0\) is a constant indicating the gain factor of the filter.

In the same approach, we obtain the transfer function for the fractional order high-pass filter, briefly labeled as \(HP,FO\):

\[
H_{HP,FO}(s) = G_0 \cdot \frac{(\tau s)^\alpha}{(\tau s)^\alpha + 1}
\]  

(2.3)

For the band-pass filter, denoted as \(BP,FO\), the resulting transfer function is the following:

\[
H_{BP,FO}(s) = G_0 \cdot \frac{(\tau s)^\beta}{(\tau s)^\alpha + 1}
\]  

(2.4)

which describes an assymetric band-pass filter with \(0 < \beta < \alpha < 1\) [30].

The transfer function is a mathematical description of the frequency domain connection between the input and output signals, used to create inverse filters with specified distortion characteristics, allowing for accurate correction of signal aberrations. Using the mother functions of conventional filters as a starting point, their inverse equivalents are created as follows:

\[
H_{INV} = (H)^{-1}
\]

This is applicable to fractional order transfer functions likewise: \(H_{INV,FO} = (H_{FO})^{-1}\).
The transfer function for the fractional-order inverse low-pass filter, or ILP,FO, is:

\[
H_{ILP,FO}(s) = G_0 \cdot [(\tau s)^a + 1]
\]  

(2.5)

Accordingly for the fractional-order inverse high-pass filter, termed briefly as IHP,FO:

\[
H_{IHP,FO}(s) = G_0 \cdot \frac{(\tau s)^a + 1}{(\tau s)^a}
\]

(2.6)

and for the fractional-order inverse band-pass filter, simply denoted as IBP,FO:

\[
H_{IBP,FO}(s) = G_0 \cdot \frac{(\tau s)^a + 1}{(\tau s)^\beta}
\]

(2.7)

### 2.3 Fractional-order frequency domain characteristics

Setting \( s = j\omega \) in Eq. (2.1), the resulting expressions for the gain and phase responses are:

\[
H_{LP}(j\omega) = \frac{1}{1 + j \left( \frac{\omega}{\omega_0} \right)}
= \frac{1}{1 + j\omega\tau}
\]

(2.8)

\[
| H_{LP}(\omega) | = \frac{1}{\sqrt{1 + (\omega\tau)^2}}
\]

(2.9)

Accordingly, when substituting the Laplace variable \( s \) with \( s^\alpha \) shown in Eq. (2.10), after computations we obtain the gain and phase responses for the low-pass fractional-order filter:

\[
s^\alpha = \omega^\alpha \left[ \cos \left( \frac{\alpha\pi}{2} \right) + j \sin \left( \frac{\alpha\pi}{2} \right) \right]
\]

(2.10)

\[
H_{LP,FO}(s) = G_0 \cdot \frac{1}{(\tau s)^a + 1} \iff H_{LP,FO}(\omega) = G_0 \cdot \frac{1}{\tau^\alpha \omega^\alpha \left[ \cos \left( \frac{\alpha\pi}{2} \right) + j \sin \left( \frac{\alpha\pi}{2} \right) \right] + 1} =
\]

\[
= G_0 \cdot \frac{1}{(\omega\tau)^\alpha \cos \left( \frac{\alpha\pi}{2} \right) + 1} + j (\omega\tau)^\alpha \sin \left( \frac{\alpha\pi}{2} \right)
\]

(2.11)
So the gain response is calculated:

\[
| H_{LP,FO}(\omega) | = G_0 \cdot \frac{\sqrt{1^2}}{\sqrt{\left[(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + 1\right]^2 + \left((\omega \tau)^\alpha \sin \left(\frac{\alpha \pi}{2}\right)\right)^2}}
\]

\[
= G_0 \cdot \frac{1}{\sqrt{(\omega \tau)^{2\alpha} \cos^2 \left(\frac{\alpha \pi}{2}\right) + 2(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + 1 + (\omega \tau)^{2\alpha} \sin^2 \left(\frac{\alpha \pi}{2}\right)}}
\]

\[
= G_0 \cdot \frac{1}{\sqrt{(\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + 1}}
\]

The phase response is estimated using formula (2.13) with the values from Eq.(2.11):

\[
\angle H(\omega) = \tan^{-1} \left( \frac{Im}{Real} \right)_{\text{numerator}} - \tan^{-1} \left( \frac{Im}{Real} \right)_{\text{denominator}} 
\]

(2.13)

So for the fractional-order low-pass filter the phase response, with \( \tau = \frac{1}{\omega_0} \), is:

\[
\angle H_{LP,FO}(\omega) = \tan^{-1} \left( \frac{0}{1} \right) - \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left(\frac{\alpha \pi}{2}\right)}{(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + 1} \right] =
\]

\[
= 0 - \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left(\frac{\alpha \pi}{2}\right)}{(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + 1} \right] = -\tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left(\frac{\alpha \pi}{2}\right)}{(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + 1} \right]
\]

which means
2.3 Fractional-order frequency domain characteristics

\[ \angle H_{LP,FO}(\omega) \big|_{\omega \to 0} = 0 \]
\[ \angle H_{LP,FO}(\omega) \big|_{\omega \to \infty} = -\alpha \cdot \frac{\pi}{2} \]

so the phase response range is \([0, -\frac{\alpha \pi}{2}]\) while the transition slope is \(-20 \cdot \alpha \, dB/dec.\)

The half-power frequency indicates the frequency at which the filter’s maximum gain changes significantly (i.e. the frequency at which a drop by 3 \(dB\) from the maximum gain value is observed, or a rise by 3 \(dB\) from the minimum gain value for Inverse filters) [2]. For Integer-order filters, the half-power frequency is always equal to the pole frequency. For non-integer order filters, it is dependent on the filter’s order, as demonstrated in Eq.(2.14) [18], therefore, scaling of the half-power frequency can be performed by modifying the order \(\alpha\). This can prove beneficial in many cases [31].

\[ \omega_{h_{LP,FO}} = \omega_0 \cdot \left[ \sqrt{1 + \cos^2 \left( \frac{\alpha \pi}{2} \right)} - \cos \left( \frac{\alpha \pi}{2} \right) \right]^{1/\alpha} \quad (2.14) \]

In the same approach, we obtain the gain and phase responses of Eqs.(2.15)-(2.16) for the fractional-order high-pass filter. First we substitute the Laplacian operator with its fractional equivalent:

\[ H_{HP,FO}(s) = G_0 \cdot \frac{(\tau s)^a}{(\tau s)^a + 1} \Leftrightarrow H_{LP,FO}(\omega) = G_0 \cdot \frac{(\omega \tau)^a \cos \left( \frac{\alpha \pi}{2} \right) + j (\omega \tau)^a \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^a \cos \left( \frac{\alpha \pi}{2} \right) + 1 + j (\omega \tau)^a \sin \left( \frac{\alpha \pi}{2} \right)} \]

So the gain response is calculated accordingly:

\[ |H_{HP,FO}(\omega)| = G_0 \cdot \frac{(\omega \tau)^a}{\left[ 1 + (\omega \tau)^{2a} + 2 (\omega \tau)^a \cos \left( \frac{\alpha \pi}{2} \right) \right]^{\frac{1}{2}}} \quad (2.15) \]

The transition from the stop-band to the pass-band has a slope of \(+20 \cdot \alpha \, dB/dec.\)
As for the phase response:

$$\angle H_{HP, FO}(\omega) = \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right)} \right] - \tan \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right]$$

$$= \tan^{-1} \left[ \tan \left( \frac{\alpha \pi}{2} \right) \right] - \tan \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right]$$

and so

$$\angle H_{HP, FO}(\omega) = \frac{\alpha \pi}{2} - \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right]$$

(2.16)

where

$$\angle H_{HP, FO}(\omega) \big|_{\omega \to 0} = \alpha \cdot \frac{\pi}{2}$$

$$\angle H_{HP, FO}(\omega) \big|_{\omega \to \infty} = 0$$

so the phase ranges between $$\left[ \frac{\alpha \pi}{2}, 0 \right]$$.

The half-power frequency for this type of filter is described in Eq.(2.17). As opposed to the low-pass filter, $$\omega_{h_{HP, FO}}$$ is located at a relatively higher frequency than that of the pole frequency.

$$\omega_{h_{HP, FO}} = \omega_0 \cdot \left[ \sqrt{1 + \cos^2 \left( \frac{\alpha \pi}{2} \right) + \cos \left( \frac{\alpha \pi}{2} \right)} \right]^\frac{1}{\beta}$$

(2.17)

The same procedure is repeated for the fractional-order band-pass filter, concluding to Eqs.(3.13)-(2.19)

$$\left| H_{BP, FO}(\omega) \right| = G_0 \cdot \frac{(\omega \tau)^\beta}{\left[ 1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right]^\frac{1}{2}}$$

(2.18)
2.3 Fractional-order frequency domain characteristics

\[ \angle H_{BP,FO}(\omega) = \frac{\beta \pi}{2} - \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right] \]  

(2.19)

This is in fact an asymmetric band-pass filter with a slope equal to \(+20 \cdot \beta \) dB/dec in the low-frequency range and equal to \(-20 \cdot (\alpha - \beta) \) dB/dec in the high frequency range. The phase angle lies within the range \( \left[ \frac{+\beta \cdot \pi}{2}, \frac{-(\alpha - \beta) \cdot \pi}{2} \right] \).

Peak frequency (\( \omega_{\text{peak}} \)) refers to the frequency at which the frequency response reaches its highest or minimum point. Solving the equation \( \frac{d}{d\omega} |H_{BP,FO}(\omega)|_{\omega=\omega_{\text{peak}}} = 0 \) yields the related lower and upper half-power frequencies (\( \omega_{h,\text{low}}, \omega_{h,\text{high}} \)) by solving the following non-linear equation: \( |H_{BP,FO}(\omega)|_{\omega=\omega_{h,\text{low}}} = \omega_{h,\text{high}} = 1/\sqrt{2} |H_{BP,FO}(\omega)|_{\omega=\omega_{\text{peak}}} \). The maximum gain (\( G_{\text{max}} \)) is determined by \( |H_{BP,FO}(\omega)|_{\omega=\omega_{\text{peak}}} = 0 \). The filter’s bandwidth is equal to \( \omega_{h,\text{high}} - \omega_{h,\text{low}} \).

For the inverse filters, the same procedure is conducted, resulting to the following:

For the fractional-order inverse low-pass filter we obtain:

\[ |H_{ILP,FO}(\omega)| = G_0 \cdot \left[ 1 + (\omega \tau)^{2\alpha} + 2 (\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right]^{\frac{1}{2}} \]  

(2.20)

\[ \angle H_{ILP,FO}(\omega) = \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right] \]  

(2.21)

In the case of inverse filters, the half-power frequency indicates the frequency at which a +3dB rise from the minimum gain is observed. For the ILP,FO Eq. (2.14) applies again.
Fractional-Order Filters

For the fractional-order inverse high-pass filter:

\[
|H_{HP,FO}(\omega)| = G_0 \cdot \frac{1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right)}{(\omega \tau)^\alpha}^{\frac{1}{2}}
\]  

(2.22)

\[
\angle H_{IH,P,FO}(\omega) = \tan^{-1} \frac{(\omega \tau)^\alpha \sin \left(\frac{\alpha \pi}{2}\right)}{(\omega \tau)^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + 1} - \frac{\alpha \pi}{2}
\]  

(2.23)

As for the half-power frequency, Eq. (2.17) applies again.

The associated gain responses are the inverse of those given by Eqs. (2.12) and (2.15), resulting in slopes that are the inverse of those found in conventional low-pass and high-pass filters.

For the fractional-order inverse band-pass filter:

\[
|H_{BP,FO}(\omega)| = G_0 \cdot \frac{1 + (\omega \tau)^{2\beta} + 2(\omega \tau)^\beta \cos \left(\frac{\beta \pi}{2}\right)}{(\omega \tau)^\beta}^{\frac{1}{2}}
\]  

(2.24)

\[
\angle H_{IB,P,FO}(\omega) = \tan^{-1} \frac{(\omega \tau)^\beta \sin \left(\frac{\beta \pi}{2}\right)}{(\omega \tau)^\beta \cos \left(\frac{\beta \pi}{2}\right) + 1} - \frac{\beta \pi}{2}
\]  

(2.25)

The filter’s peak frequency ($\omega_{peak}$) and minimum gain ($G_{min}$) are determined using the same criteria as before. The lower and upper half-power frequencies are computed by altering the corresponding condition as follows:

\[
|H_{BP,FO}(\omega)|_{\omega=\omega_{low,\omega_{high}}} = \sqrt{2} \cdot |H_{BP,FO}(\omega)|_{\omega=\omega_{peak}}.
\]
Chapter 3

Power-Law Filters

3.1 Introduction

Power-Law filters constitute a different category of non-integer order filters. Their distinction from fractional order filters is the position of the exponent. More specifically, the non-integer class has the full transfer function as its base, rather than simply the Laplacian operator $s$. This indicates they have the following form $H_{PL} = (H)^\gamma$, where $0 < \gamma < 1$. The variable $\gamma$ is used for convenience reasons in the following category of Double-order Filters.

3.2 Power-Law transfer functions

To construct the power-law transfer functions of the conventional filters we will use the mother function again, as previously. For the low-pass filter, Eq. (2.1) will be used and the result is the following transfer function denoted as LP,PL meaning low-pass power-law:

$$H_{LP,PL}(s) = G_0 \left(\frac{1}{\tau s + 1}\right)^\gamma = G_0 \frac{1}{\tau s + 1}^{\gamma}$$  \hspace{1cm} (3.1)

Accordingly for the high-pass power-law filter, briefly denoted as HP,PL:

$$H_{HP,PL}(s) = G_0 \cdot \left(\frac{\tau s}{\tau s + 1}\right)^\gamma$$  \hspace{1cm} (3.2)
and for the band-pass power-law, denoted as BP,PL:

\[ H_{BP,PL}(s) = G_0 \left( \frac{\tau s}{\tau s + 1} \right)^\gamma \]  
(3.3)

The equivalent inverse filters are extracted by the previous method and they are below stated for each category:

The inverse low-pass power-law filter, briefly denoted as ILP,PL:

\[ H_{ILP,PL}(s) = G_0 \cdot (\tau s + 1)^\gamma \]  
(3.4)

The inverse high-pass power-law filter, or IHP,PL:

\[ H_{IHP,PL}(s) = G_0 \cdot \left( \frac{\tau s + 1}{\tau s} \right)^\gamma \]  
(3.5)

The inverse band-pass power-law filter, briefly denoted as IBP,PL:

\[ H_{IBP,PL}(s) = G_0 \cdot \left( \frac{\tau s + 1}{\tau s} \right)^\gamma \]  
(3.6)

### 3.3 Power-Law frequency domain characteristics

Following the same procedure as in Chapter 2.3 we obtain the following expressions for the gain and phase response of all filter cases accordingly.

For the low-pass power-law filter, briefly denoted as LP,PL:

\[ | H_{LP,PL}(\omega) | = G_0 \cdot \left( \frac{\sqrt{1}}{\sqrt{1 + (\omega \tau)^2}} \right)^\gamma = G_0 \cdot \frac{1}{\left[1 + (\omega \tau)^2\right]^{\gamma/2}} \]  
(3.7)

The variables \( G_0 \) and \( \tau \) have the same correspondence as in fractional-order filters. Setting \( s = j\omega \):

\[ H_{LP,PL}(j\omega) = \left( \frac{1}{1 + j\left( \frac{\omega}{\omega_0} \right)\tau} \right)^\gamma = \left( \frac{1}{1 + j\omega \tau} \right)^\gamma \]
3.3 Power-Law frequency domain characteristics

using Eq. (2.13) we obtain:

\[ \angle H_{LP,PL}(\omega) = -\gamma \cdot \tan^{-1}(\omega \tau) \] (3.8)

while the half-power frequency

\[ \omega_h = \omega_0 \sqrt{2^{1/\gamma} - 1} \] (3.9)

The value of the exponential \( \gamma \) is a parameter of the function, thus, scaling of the half-power frequency is again feasible.

The slope of the transition between the filter’s two distinct bands is \(-20 \cdot \gamma \text{ dB/dec}\) and the range of the phase response is \(\left[0, \frac{-\gamma \pi}{2}\right]\) [32].

For the high-pass power-law filter, briefly denoted as HP,PL:

\[ | H_{HP,PL}(\omega)| = G_0 \cdot \frac{(\omega \tau)^\gamma}{\left[1 + (\omega \tau)^2\right]^{\gamma/2}} \] (3.10)

\[ \angle H_{HP,PL}(\omega) = \gamma \cdot \left[\frac{\pi}{2} - \tan^{-1}(\omega \tau)\right] \] (3.11)

while the half-power frequency

\[ \omega_h = \frac{\omega_0}{\sqrt{2^{1/\gamma} - 1}} \] (3.12)

The slope of the transition between the filter’s two distinct bands is \(+20 \cdot \gamma \text{ dB/dec}\) and the range of the phase response is \(\left[\frac{\gamma \pi}{2}, 0\right]\).

For the band-pass power-law filter, briefly denoted as BP,PL:

\[ | H_{BP,PL}(\omega)| = G_0 \cdot \frac{(\omega \tau)^{\beta \gamma}}{\left[1 + (\omega \tau)^2\right]^{\gamma/2}} \] (3.13)

\[ \angle H_{BP,PL}(\omega) = \beta \cdot \gamma \cdot \frac{\pi}{2} - \gamma \cdot \tan^{-1}(\omega \tau) \] (3.14)

This illustrates an asymmetric band-pass filter with \(0 < \beta, \gamma < 1\). The transition slope is \(+20 \cdot \beta \cdot \gamma \text{ dB/dec}\) in the low frequency range and \(-20 \cdot (1 - \beta) \cdot \gamma \text{ dB/dec}\) in the high frequency region.
Power-Law Filters

The phase angle is between the range \[+\frac{\beta \cdot \gamma \cdot \pi}{2}, -\frac{(1 - \beta) \cdot \gamma \cdot \pi}{2}\], while the half-power frequency is calculated through the same procedure as before.

For the inverse low-pass power-law filter, briefly denoted as ILP,PL:

\[
| H_{ILP,PL}(\omega) | = G_0 \left[ 1 + (\omega \tau)^2 \right]^{\gamma/2} 
\] (3.15)

\[
\angle H_{ILP,PL}(\omega) = \gamma \cdot \tan^{-1}(\omega \tau)
\] (3.16)

The half-power frequency is given again by Eq. (3.9), as in the case of LP,FO and ILP,FO filters.

For the inverse high-pass power-law filter, briefly denoted as IHP,PL:

\[
| H_{IHP,PL}(\omega) | = G_0 \cdot \frac{\left[ 1 + (\omega \tau)^2 \right]^{\gamma/2}}{(\omega \tau)^{\gamma}}
\] (3.17)

\[
\angle H_{IHP,PL}(\omega) = \gamma \cdot \left[ \tan^{-1}(\omega \tau) - \frac{\pi}{2} \right]
\] (3.18)

The half-power frequency is given by Eq. (3.12), accordingly.

For the inverse band-pass power-law filter, briefly denoted as IBP,PL:

\[
| H_{IBP,PL}(\omega) | = G_0 \cdot \left[ \frac{1 + (\omega \tau)^2 + 2 (\omega \tau)^{\alpha} \cos \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^{\beta}} \right]^{\gamma/2}
\] (3.19)

\[
\angle H_{IBP,PL}(\omega) = \gamma \cdot \tan^{-1}(\omega \tau) - \beta \cdot \frac{\pi}{2}
\] (3.20)

The filter’s peak frequency \((\omega_{\text{peak}})\), minimum gain \((G_{\text{min}})\) as well as the lower and upper half-power frequencies \((\omega_{h,\text{low}}, \omega_{h,\text{high}})\) are calculated using the same equations as in Chapter 2.3.
Chapter 4

Generalized-Order Filters

4.1 Introduction

Generalized-order filters combine the features of fractional-order and power-law dynamics, providing a flexible solution to a wide range of challenges. These filters are defined by two essential parameters: the fractional order $\alpha$ and the power-law exponent $\gamma$, so they are Double-order filters. Researchers and engineers can use this extra degree of freedom to tailor the filter’s behaviour to specific requirements. Bridging the gap between fractional-order and power-law filters, Generalized-order filters provide a unified framework that transcends conventional filter boundaries.

4.2 Double-order Filters

The third category of filters are double-order filters, exhibiting both fractional and power-law characteristics. Specifically, the non-integer order influences the mother function twice: once directly on the Laplacian operator $'s'$ and the second time on the entire transfer function $H$. If, for example, we consider the mother function of a low-pass filter, denoted as Eq. (2.1), the resulting function is represented by

$$H_{LP,DO} = G_0 \left( \frac{1}{(\tau s)^\alpha + 1} \right)^\gamma$$

which includes indeed two orders, $\alpha$ and $\gamma$. 

We comprehend that the fractional-order (FO) and power-law (PL) filters are subsets within the category of double-order filters. A single dual-order function, with appropriately assigned values, can result in either a FO or PL filter. Specifically, if $\gamma = 1$ and $0 < \alpha < 1$, it results in a fractional-order filter as follows:

$$H_{LP,DO}(s) = G_0 \left( \frac{1}{(\tau s)^a + 1} \right)^1 = G_0 \frac{1}{(\tau s)^a + 1} = H_{LP,FO}(s)$$

Conversely, if $\alpha = 1$ and $0 < \gamma < 1$, a power-law filter is obtained:

$$H_{LP,DO}(s) = G_0 \left( \frac{1}{(\tau s)^\gamma + 1} \right)^\gamma = G_0 \left( \frac{1}{\tau s + 1} \right)^\gamma = H_{LP,PL}(s)$$

As double-order filters serve as a superset encompassing all, we will now refer to them as generalized filters, representing the comprehensive form for each possible combination.

### 4.3 Generalized-order transfer functions

For the generalized low-pass filter, denoted as LP,GEN:

$$H_{LP,GEN}(s) = G_0 \left( \frac{1}{(\tau s)^a + 1} \right)^\gamma$$

(4.1)

For the generalized high-pass filter, denoted as HP,GEN:

$$H_{HP,GEN}(s) = G_0 \left( \frac{(\tau s)^a}{(\tau s)^a + 1} \right)^\gamma$$

(4.2)

For the generalized band-pass filter, denoted as BP,GEN:

$$H_{BP,GEN}(s) = G_0 \left( \frac{(\tau s)^\beta}{(\tau s)^a + 1} \right)^\gamma$$

(4.3)

The transfer functions of the inverse generalised filters are are easily derived from their conventional counterparts by setting $\gamma \rightarrow -\gamma$, as shown later in Tables 5.1b, 5.1c and 5.1d.
4.4 Generalized-order frequency domain characteristics

For the generalized inverse low-pass filter, denoted as \( \text{ILP,GEN} \):

\[
H_{\text{INV.LP,GEN}}(s) = G_0 \left( \frac{1}{(\tau s)^a + 1} \right)^{-\gamma} = G_0 \left[ (\tau s)^a + 1 \right]^\gamma \quad (4.4)
\]

For the generalized inverse high-pass filter, denoted as \( \text{IHP,GEN} \):

\[
H_{\text{INV.HP,GEN}}(s) = G_0 \left( \frac{(\tau s)^a}{(\tau s)^a + 1} \right)^{-\gamma} = G_0 \left( \frac{(\tau s)^a + 1}{(\tau s)^a} \right)^\gamma \quad (4.5)
\]

For the generalized inverse band-pass filter, denoted as \( \text{IBP,GEN} \):

\[
H_{\text{INV.BP,GEN}}(s) = G_0 \left( \frac{(\tau s)^\beta}{(\tau s)^\beta + 1} \right)^{-\gamma} = G_0 \left( \frac{(\tau s)^\beta + 1}{(\tau s)^\beta} \right)^\gamma \quad (4.6)
\]

4.4 Generalized-order frequency domain characteristics

The corresponding gain responses are as follows:

For the generalized low-pass filter

\[
| H_{\text{LP,GEN}}(\omega) | = G_0 \cdot \left\{ \frac{1}{\left[ 1 + (\omega \tau)^{2a} + 2 (\omega \tau)^a \cos \left( \frac{\alpha \pi}{2} \right) \right]^{\frac{1}{2}}} \right\}^\gamma \quad (4.7)
\]

\[
\angle H_{\text{LP,GEN}}(\omega) = -\gamma \cdot \tan^{-1} \left[ \frac{(\omega \tau)^a \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^a \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right] \quad (4.8)
\]

\[
\omega_{\text{h,LP,GEN}} = \omega_0 \left[ \sqrt{2^{1/\gamma} - \sin^2 \left( \frac{\alpha \pi}{2} \right) - \cos \left( \frac{\alpha \pi}{2} \right)} \right]^{1/\alpha} \quad (4.9)
\]

and the slope being \(-20 \cdot \alpha \cdot \gamma \text{ dB/dec.}\)
For the generalized high-pass filter:

\[
|H_{\text{HP,GEN}}(\omega)| = G_0 \cdot \left\{ \frac{(\omega \tau)^{\alpha}}{\left[1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^{\alpha} \cos\left(\frac{\alpha \pi}{2}\right)\right]^2} \right\}^\gamma \tag{4.10}
\]

\[
\angle H_{\text{HP,GEN}}(\omega) = \gamma \cdot \left\{ \frac{\alpha \pi}{2} - \tan^{-1}\left[\frac{(\omega \tau)^{\alpha} \sin\left(\frac{\alpha \pi}{2}\right)}{(\omega \tau)^{\alpha} \cos\left(\frac{\alpha \pi}{2}\right) + 1}\right]\right\} \tag{4.11}
\]

\[
\omega_{h,\text{HP,GEN}} = \frac{\omega_0}{\sqrt{2^{1/\gamma} - \sin^2\left(\frac{\alpha \pi}{2}\right) - \cos\left(\frac{\alpha \pi}{2}\right)^{1/\alpha}}}, \tag{4.12}
\]

and slope \(+20 \cdot \alpha \cdot \gamma \, \text{dB/dec}\).

For the generalized band-pass filter:

\[
|H_{\text{BP,GEN}}(\omega)| = G_0 \cdot \left\{ \frac{(\omega \tau)^{\beta}}{\left[1 + (\omega \tau)^{2\beta} + 2(\omega \tau)^{\beta} \cos\left(\frac{\beta \pi}{2}\right)\right]^2} \right\}^\gamma \tag{4.13}
\]

\[
\angle H_{\text{BP,GEN}}(\omega) = \gamma \cdot \left\{ \frac{\beta \pi}{2} - \tan^{-1}\left[\frac{(\omega \tau)^{\beta} \sin\left(\frac{\beta \pi}{2}\right)}{(\omega \tau)^{\beta} \cos\left(\frac{\beta \pi}{2}\right) + 1}\right]\right\} \tag{4.14}
\]

The peak frequency and the maximum gain are calculated using the same formulas as in Chapter 2, so \(\frac{d}{d\omega}|H_{\text{BP,GEN}}(\omega)|_{\omega=\omega_{\text{peak}}} = 0\) and \(|H_{\text{BP,GEN}}(\omega)|_{\omega=\omega_{\text{peak}}} = 0\). The lower and upper half-power frequencies (\(\omega_{h,\text{low}}, \omega_{h,\text{high}}\)) are derived through the solutions of the non-linear equation: \(|H_{\text{BP,GEN}}(\omega)|_{\omega=\omega_{h,\text{low}}, \omega_{h,\text{high}}} = \frac{1}{\sqrt{2}} G_{\text{max}}\) and, therefore, the bandwidth of the filter is calculated again according to the formula: \(BW = \omega_{h,\text{high}} - \omega_{h,\text{low}}\). The slope of the transition form the stop-band to the pass-band of the filter in the low-frequency range is equal to \(+20\beta \cdot \gamma \, \text{dB/dec}\), while in the high-frequency range the transition to the stop-band is performed with a slope \(-20(\alpha - \beta) \cdot \gamma \, \text{dB/dec}\).
4.4 Generalized-order frequency domain characteristics

For the generalized inverse low-pass filter:

\[
| H_{\text{LP,GEN}}(\omega) | = G_0 \cdot \left\{ 1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right\}^{\frac{1}{2}} \gamma
\] (4.15)

\[
\angle H_{\text{ILP,GEN}}(\omega) = \gamma \cdot \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right]
\] (4.16)

As explained in Chapter 2, in the case of inverse filters, the half-power frequency indicates the frequency at which a +3 dB rise from the minimum gain is observed. For the ILP,GEN Eq. (4.9) applies again.

For the generalized inverse high-pass filter:

\[
| H_{\text{HHP,GEN}}(\omega) | = G_0 \cdot \left\{ 1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right\}^{\frac{1}{2}} \gamma
\] (4.17)

\[
\angle H_{\text{IHP,GEN}}(\omega) = \gamma \cdot \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right] - \frac{\alpha \pi}{2}
\] (4.18)

As for the half-power frequency, Eq. (4.12) applies again.

For the generalized inverse band-pass filter:

\[
| H_{\text{IBP,GEN}}(\omega) | = G_0 \cdot \left\{ 1 + (\omega \tau)^{2\alpha} + 2(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right\}^{\frac{1}{2}} \gamma
\] (4.19)

\[
\angle H_{\text{IBP,GEN}}(\omega) = \gamma \cdot \tan^{-1} \left[ \frac{(\omega \tau)^\alpha \sin \left( \frac{\alpha \pi}{2} \right)}{(\omega \tau)^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + 1} \right] - \frac{\beta \pi}{2}
\] (4.20)
The same procedure as the previous chapters applies yet again. The filter’s peak frequency \( \omega_{\text{peak}} \) and minimum gain \( G_{\text{min}} \) are determined using the same criteria as before. The lower and upper half-power frequencies \( \omega_{k,l}, \omega_{k,h} \) are computed by altering the corresponding condition as follows:

\[
|H_{IBP,\text{GEN}}(\omega)|_{\omega=\omega_{k,\text{low}},\omega_{k,\text{high}}} = \sqrt{2}|H_{IBP,\text{GEN}}(\omega)|_{\omega=\omega_{\text{peak}}}.
\]

Generalized Filters offer three degrees of freedom \( \alpha, \beta, \text{ and } \gamma \), so they provide greater flexibility in adjusting different characteristics individually, such as half-power frequency, slope of transition, peak frequency, maximum gain and bandwidth. In contrast, fractional-order and power-law filters, with their single order \( \alpha \) or \( \gamma \), lack the ability to independently control these parameters, limiting their adaptability in adjusting specific filter characteristics.
Chapter 5

Approximation of generalized non-integer order filters

5.1 Introduction

To implement non-integer order filters at circuit level, an intermediate step of estimating their transfer functions is required. This chapter presents mathematical approaches and procedures for approximating the transfer functions of fractional-order, power-law and double-order low-pass filters. To implement the estimated transfer functions, the Inverse-Follow-the-Leader-Feedback (IFLF) multiple-feedback filter topology is implemented and the design process is studied in detail. Finally, the gain and phase responses of the ideal and approximated transfer functions are presented in MATLAB.

Conventional approximation methods such as Continued Fraction Expansion (CFE), Oustaloup [7] and Matsuda [10, 11] can simulate the behaviour of the Laplacian operator $s^\alpha$ to implement fractional-order filters [33, 34, 2, 35, 36]. However, when dealing with more complex filters, such as power-law and double-order filters, these approaches are not suitable due to the fact that multiple parameters are required. To approximate these formulas, complicated approaches are required that reproduce the entire transfer function.

Existing research has identified two approximation strategies for such applications: curve fitting of frequency response data [37–39] and the Padé approximation tool [40, 41]. The former involves fitting a curve to the function’s frequency response data, while the latter employs the Padé approximation tool to extend the function asymptotically around a specified frequency. Both methods are effective approximation tools, producing a transfer function characterized by a ratio of integer-order polynomials. This Thesis employs the curve fitting approach to evaluate the examined filters.
The proposed approximation method is fully supported by MATLAB software in cooperation with the FOMCON (Fractional-Order Modeling and Control) toolbox [42], used to accurately perform the required calculations.

5.2 Curve fitting approximation technique

MATLAB code analysis

The curve fitting based approximation technique is a method used to generate transfer functions that realize the intended frequency characteristics of fractional-order prototype filters, such as slope and cut-off frequencies. This approach fits the frequency response data of the magnitude response functions to a minimum-phase state-space model and relies on estimating the transfer function using the Sanathanan-Koerner (SK) [43] least-square iterative method [44, 45].

The transfer function that results from this process will take the form of the following equation

\[ H(s) = \frac{A_n s^n + A_{n-1} s^{n-1} + \cdots + A_1 s + A_0}{B_n s^n + B_{n-1} s^{n-1} + \cdots + B_1 s + B_0}, \]  

(5.1)

where \( A_i \) and \( B_j \) \((i = 0, 1...n, j = 0, 1...n)\) are positive and real coefficients; and \( n \) represents the approximation order.

To achieve the approximate function for all types of filters, a modification of the MATLAB code from [38, 39, 46] is needed. The final code is presented in Appendix A.1 and executes the following procedure:

• First, the user-defined parameters are declared. These include the coefficients used in the non-integer order transfer function, the gain factor and the order of approximation for the fit. In addition, the unaltered characteristics are defined, namely the cut-off frequency of the system, the value of the capacitors used in the implementation, the bias current and two parameters related to the MOS transistor. Internal calculations are then performed, including the time constant \( \tau = \frac{1}{\omega_0} \) and the assignment of values to the minimum and maximum frequency variables used to create a logarithmically spaced frequency vector between them.
5.3 Design examples

- Next, the ideal fractional order intermediate transfer function is defined using the \texttt{fotf} function and its frequency response is calculated at the specified frequency vector.

- The calculated frequency response is then raised to the power of $\gamma$ and multiplied by $G_0$. This creates the exponential based on the entire function as well as the multiplication by the desired gain factor.

- The adjusted frequency response is converted into a Frequency Response Data (FRD) object using the \texttt{frd} function. This object is useful for subsequent fitting procedures.

- The \texttt{fitfrd} function is used to fit the FRD data to a transfer function of the specified approximation order.

- Then, \texttt{minreal} is applied to obtain the minimum realisation of the fitted transfer function (H\_FIT\_FRD). This step contributes to reducing the order of the transfer function while maintaining its key dynamics.

- The \texttt{tfdata} function extracts the numerator $A_i$ and denominator $B_j$ coefficients.

- A \texttt{for loop} then extracts the time constants (tau), transconductance ($g_m$) and theoretical bias currents of the OTAs (Io) as well as the scaling factors (G) for each current, data that will be used in the implementation in Cadence.

- A computation displays the Bode plot comparisons (gain and phase) for both the ideal and fitted transfer functions as well as their errors. It is important to note that for the phase response Bode plot, the phase is multiplied by gamma, as explained in the theory of the previous chapters.

5.3 Design examples

Tables 5.1 provide the exponent values evaluated for each filter category. The gain values for the band-pass and inverse band-pass filters were selected based on the gain response graph from MATLAB. Specifically, the values were chosen at the point where the graph displayed a maximum and a minimum critical point at 0 $dB$ for each of the cases.

It appears that the exponents are the same for the conventional and inverse filters, except for $\gamma$, which is the opposite in the latter case, as explained in Chapter 3.
### Approximation of generalized non-integer order filters

#### Table 5.1 Exponent and gain factor values for conventional filters and their inverse counterparts

<table>
<thead>
<tr>
<th>Filter</th>
<th>Gain</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>Filter</th>
<th>Gain</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>Filter</th>
<th>Gain</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>1</td>
<td>0.3</td>
<td>0</td>
<td>1</td>
<td>LP</td>
<td>1</td>
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<tr>
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<td>0.3</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
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<td>0.3</td>
<td>1</td>
<td>BP</td>
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<td>1</td>
<td>0.3</td>
<td>0.5</td>
<td>BP</td>
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<td>0.3</td>
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<td>0</td>
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<td>ILP</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-0.3</td>
<td>ILP</td>
<td>1</td>
<td>0.3</td>
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<tr>
<td>IHP</td>
<td>1</td>
<td>0.3</td>
<td>0.3</td>
<td>-1</td>
<td>IHP</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-0.3</td>
<td>IHP</td>
<td>1</td>
<td>0.3</td>
<td>0.3</td>
<td>-0.3</td>
</tr>
<tr>
<td>IBP</td>
<td>0.549</td>
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<td>0.3</td>
<td>-1</td>
<td>IBP</td>
<td>0.86</td>
<td>1</td>
<td>0.3</td>
<td>-0.5</td>
<td>IBP</td>
<td>0.742</td>
<td>0.5</td>
<td>0.3</td>
<td>-0.5</td>
</tr>
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</table>

After executing the code of Appendix A.1, we result to the numerator ($A_i$) and denominator ($B_j$) values of Tables 5.2 to 5.7 for each type of filter approximated transfer function and we get the following gain response graphs of Figs. 5.1, 5.2 and 5.3 from MATLAB for the approximated and ideal transfer functions, given by dashes.

The graphs indicate that low-pass fractional-order filters show a greater decrease in signal strength at lower frequencies than at higher frequencies. In contrast, power-law filters exhibit a more abrupt response at higher frequencies, sharply decreasing signal amplitude while allowing full gain initially. On the other hand, generalized filters provide a smoother response across the entire frequency spectrum, gradually reducing signal strength and cutting off less abruptly.

For high-pass filters, a similar behaviour is observed as in low-pass filters, but with an inversion in frequency, aligning with their names.

In the case of band-pass filters, fractional-order designs show greater attenuation at low frequencies and increased pass-through at higher frequencies, while power-law filters display the opposite behavior. Generalized band-pass filters, however, provide more consistent signal gain across the entire frequency range.
5.3 Design examples

### Table 5.2 Fractional-order conventional filters numerator and denominator values

<table>
<thead>
<tr>
<th>Filter</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>8.278·10^{11}</td>
<td>1.084·10^{10}</td>
<td>4.646·10^{7}</td>
<td>2.1722</td>
<td>4.646·10^{4}</td>
</tr>
<tr>
<td>HP</td>
<td>1.722·10^{11}</td>
<td>1.084·10^{10}</td>
<td>4.646·10^{7}</td>
<td>0.8278</td>
<td>2.1722·10^{4}</td>
</tr>
<tr>
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</table>

### Table 5.3 Power-law conventional filters numerator and denominator values

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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>LP</td>
<td>5.208·10^{7}</td>
<td>9.511·10^{6}</td>
<td>2.905·10^{12}</td>
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<td>1.418·10^{11}</td>
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<td>BP</td>
<td>7.272·10^{12}</td>
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<td>2.905·10^{12}</td>
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### Table 5.4 Generalized conventional filters numerator and denominator values

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<tr>
<td>LP</td>
<td>3.204·10^{12}</td>
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<tr>
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<td>1</td>
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<tr>
<td>BP</td>
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<td>5.641·10^{7}</td>
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Table 5.5 Fractional-order inverse filters numerator and denominator values

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<th>$A_2$</th>
<th>$A_1$</th>
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<td>$8.332 \cdot 10^{11}$</td>
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<tr>
<td></td>
<td>IHP</td>
<td>1.229</td>
<td>$2.664 \cdot 10^4$</td>
<td>$4.921 \cdot 10^7$</td>
<td>$1.058 \cdot 10^{10}$</td>
</tr>
<tr>
<td></td>
<td>IBP</td>
<td>1.821</td>
<td>$1.463 \cdot 10^5$</td>
<td>$1.513 \cdot 10^9$</td>
<td>$4.063 \cdot 10^{11}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ILP</td>
<td>1</td>
<td>$9.991 \cdot 10^4$</td>
<td>$7.143 \cdot 10^8$</td>
<td>$5.279 \cdot 10^{11}$</td>
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<tr>
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<td>IHP</td>
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<td>$1.688 \cdot 10^4$</td>
<td>$2.284 \cdot 10^7$</td>
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<td>IBP</td>
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<td>$1.178 \cdot 10^5$</td>
<td>$1.499 \cdot 10^9$</td>
<td>$2.717 \cdot 10^{11}$</td>
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Table 5.6 Power-law inverse filters numerator and denominator values

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<th>$A_2$</th>
<th>$A_1$</th>
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<td>$7.6e \cdot 10^5$</td>
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</tr>
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<td>IBP</td>
<td>6.144</td>
<td>$2.329 \cdot 10^5$</td>
<td>$6.764 \cdot 10^8$</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>ILP</td>
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<td>$2.313 \cdot 10^5$</td>
<td>$6.483 \cdot 10^9$</td>
<td>$3.585 \cdot 10^{14}$</td>
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<td>800.1</td>
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<td>$1.043 \cdot 10^5$</td>
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Table 5.7 Generalized inverse filters numerator and denominator values

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<th>$A_0$</th>
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</thead>
<tbody>
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<td>1.724</td>
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<td>IHP</td>
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<tr>
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<td>IBP</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ILP</td>
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<td>$6.739 \cdot 10^4$</td>
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<td>IHP</td>
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<td>$2.113 \cdot 10^4$</td>
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<td>$1.063 \cdot 10^5$</td>
<td>$1.273 \cdot 10^9$</td>
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</table>
5.3 Design examples

Fig. 5.1 Gain frequency responses of low-pass and inverse low-pass filters
Approximation of generalized non-integer order filters

Fig. 5.2 Gain frequency responses of high-pass and inverse high-pass filters
Fig. 5.3 Gain frequency responses of band-pass and inverse band-pass filters
Chapter 6

Implementation using OTAs

6.1 Introduction

In this Chapter, the Inverse Follow-the-Leader Feedback topology of non-integer filters is implemented employing Operational Transconductance Amplifiers (OTAs) as active elements. The schematic of the proposed topology is presented, along with the design of the OTA utilized for its realization. Moreover, the gain and phase frequency responses corresponding to both the approximated transfer functions and the schematic implementations are shown for comparison.

6.2 Inverse Follow-the-Leader Feedback (IFLF)

The approximate transfer function follows Eq. (6.4) for any combination of exponents’ values. This proves advantageous for implementation using conventional integer order techniques such as the Follow-the-Leader Feedback (FLF) [47] and Inverse Follow-the-Leader Feedback (IFLF) topologies.

In this thesis the IFLF topology is utilized as depicted in the Functional Block Diagram (FBD) of Figure 6.1. The associated transfer function is given by Eq. (6.1).

\[
H(s) = \frac{G_n s^n + \frac{G_{n-1}}{\tau_1} s^{n-1} + \cdots + \frac{G_1}{\tau_1 \tau_2 \cdots \tau_{n-1}} s + \frac{G_0}{\tau_1 \tau_2 \cdots \tau_n}}{s^n + \frac{1}{\tau_1} s^{n-1} + \cdots + \frac{1}{\tau_1 \tau_2 \cdots \tau_{n-1}} s + \frac{1}{\tau_1 \tau_2 \cdots \tau_n}},
\]  

(6.1)

where \(\tau_i\) (\(i = 1, 2, ..., n\)) and \(G_i\) (\(i = 0, 1, ..., n\)) are the appropriate time-constants and scaling factors, given by Eqs. (6.2) and (6.3), respectively.
Implementation using OTAs

\[ \tau_{i+1} = \frac{B_{n-i}}{B_{n-1-i}} (i = 0, \ldots, n - 1) \]  \hspace{1cm} (6.2)
\[ G_j = \frac{A_j}{B_j} (j = 0, \ldots, n) \]  \hspace{1cm} (6.3)

The aforementioned are derived by equating the coefficients in Eqs. (5.1) and (6.1).

Passive elements such as resistors and capacitors, as well as active elements like operational amplifiers (op-amps), Current Feedback Amplifiers (CFOAs), and Voltage-Controlled Current Conveyors (VCCSs), can be utilised to implement the aforementioned topologies. The selection of components should be based on the specific requirements of the application. In this thesis, we will use OTAs as they are electronically tunable and allow for easy system modifications. Additionally, full control over the implementation can be achieved solely through the adjustment of DC currents and scaling factors. Thus, all proposed transfer functions can be obtained with the same circuit by adjusting the bias current values in each integration and summation step.

6.3 OTA-C topology

In this thesis, we have chosen the fourth order of approximation for the IFLF topology. By setting \( n = 4 \) in Eq. (6.4) we obtain the transfer function given in Eq. (6.4) below. Figure 6.2 illustrates the associated FBD.

\[ H(s) = \frac{A_4 s^4 + A_3 s^3 + A_2 s^2 + A_1 s + A_0}{B_4 s^4 + B_3 s^3 + B_2 s^2 + B_1 s + B_0}, \]  \hspace{1cm} (6.4)
6.3 OTA-C topology

The corresponding OTA-C structure is provided in Fig. 6.3 [48]. The basic building blocks for implementing this structure are illustrated in Figs. 6.4. The output voltage expressions of the structures in these figures are:

\[ U_{\text{out}} = U_{\text{in}1} + G_i U_{\text{in}2} \quad \text{and} \quad U_{\text{out}} = \frac{g_{\text{m}i}}{C_i} (U_{\text{in}1} - U_{\text{in}2} + G_i U_{\text{in}3}) \]

which correspond to a weighted summation and a weighted integration stage. Combining them on a larger scale results in the circuit tested in this chapter.

**Fig. 6.4** OTA based implementation of the basic operations in Fig. 2 (a) weighted summation, and (b) weighted integration.
The small-signal transconductance parameter $g_m$ offers electronic tunability and is calculated using the expression:

$$g_m = \frac{5I_B}{(9n_sV_T)} \quad (6.5)$$

where $n_s$ refers to the slope factor of a MOS transistor in the subthreshold region ($1 < n_s < 2$), $V_T \simeq 26 \text{ mV}$ at $27^\circ C$ is the thermal voltage and $I_B$ is the DC bias current.

The scaling factors are controlled by the corresponding bias currents of the transconductors. The time constants of the topology in Fig. 6.3 are given by the expression:

$$\tau_i = \frac{C_i}{g_{m_i}}, (i = 1, 2, \ldots, n) \quad (6.6)$$

The transconductance of each OTA is calculated by equating Eq. (6.2) with Eq. (6.6) and solving for $g_{m_i}$. To determine the final values of the transconductances, the capacitor values must be determined. In this implementation, and in alignment with the associated code, the capacitors are set at 100 $pF$. The corresponding $g_m$ for each OTA and filter category are presented in Tables 6.1a and 6.1b.

The transistor level structure of the selected OTA is shown in Fig.6.5, while the aspect ratios of the MOS transistors are given in Table 6.2.

The theoretical values of the bias currents for the OTAs in the sub-threshold region can be calculated by Eq. (6.5) taking the form of the following equation $I_0(i) = \frac{9n_sV_T}{5g_{m_i}}$. 

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### 6.3 OTA-C topology

#### Table 6.1

Transconductance values for each stage in the cases of low-pass, high-pass and band-pass filters functions (a) and their inverse counterparts (b)

(a)

<table>
<thead>
<tr>
<th>Filter</th>
<th>gm</th>
<th>gm1</th>
<th>gm2</th>
<th>gm3</th>
<th>gm4</th>
</tr>
</thead>
<tbody>
<tr>
<td>FO-LP</td>
<td>1 µ</td>
<td>3.228 µ</td>
<td>287.8 n</td>
<td>34.74 n</td>
<td>3.097 n</td>
</tr>
<tr>
<td>PL-LP</td>
<td>1 µ</td>
<td>8.758 µ</td>
<td>1.356 µ</td>
<td>320.6 n</td>
<td>79.27 n</td>
</tr>
<tr>
<td>GEN-LP</td>
<td>1 µ</td>
<td>4.816 µ</td>
<td>397.9 n</td>
<td>45.55 n</td>
<td>3.892 n</td>
</tr>
<tr>
<td>FO-HP</td>
<td>1 µ</td>
<td>3.228 µ</td>
<td>287.8 n</td>
<td>34.74 n</td>
<td>3.097 n</td>
</tr>
<tr>
<td>PL-HP</td>
<td>1 µ</td>
<td>126.2 µ</td>
<td>31.19 n</td>
<td>7.37 n</td>
<td>1.142 n</td>
</tr>
<tr>
<td>GEN-HP</td>
<td>1 µ</td>
<td>2.569 µ</td>
<td>219.5 n</td>
<td>25.13 n</td>
<td>2.076 n</td>
</tr>
<tr>
<td>FO-BP</td>
<td>1 µ</td>
<td>8.1 v</td>
<td>980 n</td>
<td>31.95 n</td>
<td>2.867 n</td>
</tr>
<tr>
<td>PL-BP</td>
<td>1 µ</td>
<td>2.511 µ</td>
<td>242.9 n</td>
<td>21.65 n</td>
<td>2.102 n</td>
</tr>
<tr>
<td>GEN-BP</td>
<td>1 µ</td>
<td>8.864 µ</td>
<td>1.051 µ</td>
<td>27.9 n</td>
<td>2.258 n</td>
</tr>
</tbody>
</table>

(b)

<table>
<thead>
<tr>
<th>Filter</th>
<th>gm</th>
<th>gm1</th>
<th>gm2</th>
<th>gm3</th>
<th>gm4</th>
</tr>
</thead>
<tbody>
<tr>
<td>FO-ILP</td>
<td>1 µ</td>
<td>3.228 µ</td>
<td>287.8 n</td>
<td>34.74 n</td>
<td>3.097 n</td>
</tr>
<tr>
<td>PL-ILP</td>
<td>1 µ</td>
<td>8.758 µ</td>
<td>1.356 µ</td>
<td>320.6 n</td>
<td>79.27 n</td>
</tr>
<tr>
<td>GEN-ILP</td>
<td>1 µ</td>
<td>4.816 µ</td>
<td>397.9 n</td>
<td>45.55 n</td>
<td>3.892 n</td>
</tr>
<tr>
<td>FO-IHP</td>
<td>1 µ</td>
<td>3.228 µ</td>
<td>287.8 n</td>
<td>34.74 n</td>
<td>3.097 n</td>
</tr>
<tr>
<td>PL-IHP</td>
<td>1 µ</td>
<td>126.2 µ</td>
<td>31.19 n</td>
<td>7.37 n</td>
<td>1.142 n</td>
</tr>
<tr>
<td>GEN-IHP</td>
<td>1 µ</td>
<td>2.569 µ</td>
<td>219.5 n</td>
<td>25.13 n</td>
<td>2.076 n</td>
</tr>
<tr>
<td>FO-IBP</td>
<td>1 µ</td>
<td>8.1 µ</td>
<td>980 n</td>
<td>31.95 n</td>
<td>2.867 n</td>
</tr>
<tr>
<td>PL-IBP</td>
<td>1 µ</td>
<td>2.511 µ</td>
<td>242.9 n</td>
<td>21.65 n</td>
<td>2.102 n</td>
</tr>
<tr>
<td>GEN-IBP</td>
<td>1 µ</td>
<td>8.864 µ</td>
<td>1.051 µ</td>
<td>27.9 n</td>
<td>2.258 n</td>
</tr>
</tbody>
</table>

#### Table 6.2

Aspect ratios of the MOS transistors of the OTA topology in Fig.6.5

<table>
<thead>
<tr>
<th>Transistors</th>
<th>W/L (µm/µm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mp1-Mp2</td>
<td>0.5 / 1.5</td>
</tr>
<tr>
<td>Mn1-Mn2</td>
<td>1.1 / 1.5</td>
</tr>
<tr>
<td>Mn3-Mn4</td>
<td>5.5 / 1.5</td>
</tr>
<tr>
<td>Mb1-Mb3</td>
<td>4.4 / 5.0</td>
</tr>
</tbody>
</table>
Implementation using OTAs

This provides valuable insights into the magnitude of the bias currents; however, the calculations may not precisely align with the desired transconductance $gm$, potentially compromising the overall performance of the circuit and the quality of the filters. Hence, it becomes necessary to manually fine-tune the currents to ensure optimal alignment with the ideal transfer function. Note that in this Chapter the ideal transfer function refers to the approximate transfer function derived from MATLAB.

The DC current values after fine-tuning are presented in Tables 6.3a and 6.3b for the conventional filters and their inverse counterparts, accordingly. The main bias current $I_0$ for the summation stages remains consistent at 100 $nA$ in all cases. These tables provide the integration stage bias currents $I_0(i)$ for the OTAs in the top row. The bias currents for the summation stages $I(i)$ of the OTAs in the bottom row are calculated as the product of $I_0(i)$ and the scaling factor $G$, derived from the MATLAB code in Appendix A.1. This product is of the form:

$$I(i + 1) = GI_0(4 - i) \quad \text{where} \quad (i = 0, 1, \ldots, 4)$$

meaning that the scaled current $I_1$ for the first summation stage is equal to $G_0 \cdot I_{04}$ etc. The corresponding scaling factors (G) values are presented in Tables 6.4a and 6.4b.

It’s noteworthy that in Cadence, bias currents can be expressed as a product of variables during analysis. This means that by adjusting bias currents we achieve fine-tuning, as illustrated in Fig. 6.6. These figures compare the gain responses of the implementation, to the ideal scenario, while Fig. 6.7 depicts the phase responses from Cadence for all categories, again in comparison to the ideal from MATLAB.

The simulation will use the OTA circuitry with a $\pm 0.75V$ DC bias scheme obtained from the Design Kit provided by Austria Mikro Systeme (AMS) 0.35µm CMOS process, while $\omega_0 = 1 \, krad/sec$ and $C_1 = C_2 = C_3 = C_4 = 100 \, pF$. 

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Table 6.3 Basic bias currents for realizing all types of low-pass, high-pass and band-pass filters functions (a) and their inverse counterparts (b)

(a)

<table>
<thead>
<tr>
<th>Filter</th>
<th>$I_0$</th>
<th>$I_{01}$</th>
<th>$I_{02}$</th>
<th>$I_{03}$</th>
<th>$I_{04}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>100 nA</td>
<td>160 nA</td>
<td>12.5 nA</td>
<td>1.6 nA</td>
<td>180 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>460 nA</td>
<td>70 nA</td>
<td>17 nA</td>
<td>4 nA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>340 nA</td>
<td>17 nA</td>
<td>2.2 nA</td>
<td>200 pA</td>
</tr>
<tr>
<td>HP</td>
<td>100 nA</td>
<td>300 nA</td>
<td>24 nA</td>
<td>2.2 nA</td>
<td>180 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>7.6 nA</td>
<td>1.8 nA</td>
<td>400 pA</td>
<td>60 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>180.4 nA</td>
<td>15.14 nA</td>
<td>1.78 nA</td>
<td>120 pA</td>
</tr>
<tr>
<td>BP</td>
<td>100 nA</td>
<td>490 nA</td>
<td>51 nA</td>
<td>1.7 nA</td>
<td>140 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>140 nA</td>
<td>12.5 nA</td>
<td>1.2 nA</td>
<td>110 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>550 nA</td>
<td>56 nA</td>
<td>1.5 nA</td>
<td>116 pA</td>
</tr>
</tbody>
</table>

(b)

<table>
<thead>
<tr>
<th>Filter</th>
<th>$I_0$</th>
<th>$I_{01}$</th>
<th>$I_{02}$</th>
<th>$I_{03}$</th>
<th>$I_{04}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ILP</td>
<td>100 nA</td>
<td>310 nA</td>
<td>20 nA</td>
<td>4 nA</td>
<td>320 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>700 nA</td>
<td>196.8 nA</td>
<td>40 nA</td>
<td>7 nA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>275 nA</td>
<td>19 nA</td>
<td>2.6 nA</td>
<td>230 pA</td>
</tr>
<tr>
<td>IHP</td>
<td>100 nA</td>
<td>140 nA</td>
<td>9.5 nA</td>
<td>880 pA</td>
<td>53.5 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>5 nA</td>
<td>1 nA</td>
<td>190 pA</td>
<td>22 pA</td>
</tr>
<tr>
<td></td>
<td>100 nA</td>
<td>200 nA</td>
<td>12 nA</td>
<td>1.2 nA</td>
<td>79 pA</td>
</tr>
<tr>
<td>IBP</td>
<td>100 nA</td>
<td>700 nA</td>
<td>60 nA</td>
<td>940 pA</td>
<td>53 pA</td>
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<td></td>
<td>100 nA</td>
<td>470 nA</td>
<td>26 nA</td>
<td>850 pA</td>
<td>66 pA</td>
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<tr>
<td></td>
<td>100 nA</td>
<td>600 nA</td>
<td>55 nA</td>
<td>1.1 nA</td>
<td>70 pA</td>
</tr>
</tbody>
</table>
### Table 6.4 Scaling factors for realizing all types of low-pass, high-pass and band-pass filters functions (a) and their inverse counterparts (b)

#### (a)

<table>
<thead>
<tr>
<th>Filter</th>
<th>( G_0 )</th>
<th>( G_1 )</th>
<th>( G_2 )</th>
<th>( G_3 )</th>
<th>( G_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>FO-LP</td>
<td>827 m</td>
<td>664 m</td>
<td>500 m</td>
<td>336 m</td>
</tr>
<tr>
<td></td>
<td>PL-LP</td>
<td>1</td>
<td>762.4 m</td>
<td>531.3 m</td>
<td>327.9 m</td>
</tr>
<tr>
<td></td>
<td>GEN-LP</td>
<td>943.1 m</td>
<td>879.9 m</td>
<td>805.5 m</td>
<td>712.2 m</td>
</tr>
<tr>
<td>HP</td>
<td>FO-HP</td>
<td>172.2 m</td>
<td>335.7 m</td>
<td>500 m</td>
<td>604.3 m</td>
</tr>
<tr>
<td></td>
<td>PL-HP</td>
<td>157.3 m</td>
<td>327.9 m</td>
<td>531.3 m</td>
<td>762.4 m</td>
</tr>
<tr>
<td></td>
<td>GEN-HP</td>
<td>585.1 m</td>
<td>712.2 m</td>
<td>805.5 m</td>
<td>879.9 m</td>
</tr>
<tr>
<td>BP</td>
<td>FO-BP</td>
<td>341.8 m</td>
<td>696.5 m</td>
<td>981.9 m</td>
<td>795.5 m</td>
</tr>
<tr>
<td></td>
<td>PL-BP</td>
<td>510.9 m</td>
<td>759.8 m</td>
<td>976.2 m</td>
<td>504.7 m</td>
</tr>
<tr>
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<td>GEN-BP</td>
<td>584.8 m</td>
<td>831.8 m</td>
<td>994.6 m</td>
<td>895 m</td>
</tr>
</tbody>
</table>

#### (b)

<table>
<thead>
<tr>
<th>Filter</th>
<th>( G_0 )</th>
<th>( G_1 )</th>
<th>( G_2 )</th>
<th>( G_3 )</th>
<th>( G_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ILP</td>
<td>FO-ILP</td>
<td>1.229</td>
<td>1.578</td>
<td>2.155</td>
<td>3.312</td>
</tr>
<tr>
<td></td>
<td>PL-ILP</td>
<td>1</td>
<td>1.348</td>
<td>1.996</td>
<td>3.286</td>
</tr>
<tr>
<td></td>
<td>GEN-ILP</td>
<td>1.062</td>
<td>1.141</td>
<td>1.25</td>
<td>1.418</td>
</tr>
<tr>
<td>IHP</td>
<td>FO-IHP</td>
<td>6.405</td>
<td>3.312</td>
<td>2.155</td>
<td>1.578</td>
</tr>
<tr>
<td></td>
<td>GEN-IHP</td>
<td>1.724</td>
<td>1.418</td>
<td>1.25</td>
<td>1.141</td>
</tr>
<tr>
<td>IBP</td>
<td>FO-IBP</td>
<td>3.086</td>
<td>1.496</td>
<td>1.009</td>
<td>1.243</td>
</tr>
<tr>
<td></td>
<td>PL-IBP</td>
<td>1.952</td>
<td>1.329</td>
<td>1.026</td>
<td>2.234</td>
</tr>
<tr>
<td></td>
<td>GEN-IBP</td>
<td>1.737</td>
<td>1.218</td>
<td>1.006</td>
<td>1.117</td>
</tr>
</tbody>
</table>
6.3 OTA-C topology

(a): Low-pass and Inverse Low-pass

(b): High-pass and Inverse High-pass

(c): Band-pass and Inverse Band-pass

Fig. 6.6 All types of filters’ gain frequency responses in dB
Implementation using OTAs

(a): Low-pass and Inverse Low-pass

(b): High-pass and Inverse High-pass

(c): Band-pass and Inverse Band-pass

Fig. 6.7 All types of filters’ phase frequency response in degrees
Chapter 7

Layout and Post-Layout

7.1 Introduction

The layout design of the OTA is presented, followed by the post-layout results, including time domain evaluation, Monte carlo analysis, Total Harmonic Distortion and power dissipation for indicative cases. The evaluation of the proposed system is conducted using the Cadence software analog design environment and the AMS 0.35 µm CMOS process design kit.

7.2 Layout design

The final stage involves implementing the Operational Transconductance Amplifiers (OTAs) at the layout level and verifying the accuracy of the implementation. Due to the high capacitor values involved, integrating the entire system onto a single chip is not feasible. Fig. 7.1 depicts the layout design of the OTA, with dimensions of 23.9µm × 29.3µm. Fig. 7.2 depicts the entire layout design, with dimensions of 112.8 µm × 92.15 µm, without the capacitors due to their size. The entire structure has been successfully verified and validated using Cadence design rule tools. The Design Rule Check (DRC) ensures that the design adheres to all constraints and rules provided by the foundry for the specific technology. Following the DRC, the Assura Layout Versus Schematic (LVS) tool compares the layout design with the schematic to verify their alignment. Finally, the Quantus QRC is used to extract parasitic effects.

Not that by intentionally limiting the metal layers to Metal 2, the thesis demonstrates a dedication to simplified design approaches and resource optimization, which is consistent with contemporary integrated circuit design principles.
Fig. 7.1 Layout design of the OTA

Fig. 7.2 Layout design of the circuit, not including the capacitors
7.3 Simulation results

The simulation will use the OTA circuitry with a \( \pm 0.75 \) V dc bias scheme obtained from the Design Kit provided by Austria Mikro Systeme (AMS) 0.35\( \mu \)m CMOS process, while \( \omega_0 = 1 \) krad/sec and \( C_1 = C_2 = C_3 = C_4 = 100 \) pF. The obtained post-layout gain responses of the filters under study are presented in Fig. 7.3. Furthermore, the cut-off frequencies for the cases of the low-pass, high-pass, inverse low-pass and inverse high-pass filters are summarized in Tables 7.1a and 7.1b. Overall, it can be seen that the final results are close to the theoretically expected.

![Simulation and Post-Layout gain responses for the cases of low-pass and inverse low-pass (a), high-pass and inverse high-pass (b) and band-pass and inverse band-pass filters (c)](image)

Fig. 7.3 Simulation and Post-Layout gain responses for the cases of low-pass and inverse low-pass (a), high-pass and inverse high-pass (b) and band-pass and inverse band-pass filters (c)
Table 7.1 Cut-off frequencies for all categories of low-pass, inverse low-pass, (a) high-pass and inverse high-pass filters (b)

<table>
<thead>
<tr>
<th>Filter</th>
<th>freq.</th>
<th>Filter</th>
<th>freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) LP</td>
<td></td>
<td>(b) HP</td>
<td></td>
</tr>
<tr>
<td>FO-LP</td>
<td>396.46 rad/s</td>
<td>FO-HP</td>
<td>2.29 krad/s</td>
</tr>
<tr>
<td>PL-LP</td>
<td>3.141 krad/s</td>
<td>PL-HP</td>
<td>345.2 rad/s</td>
</tr>
<tr>
<td>GEN-LP</td>
<td>27.96 krad/s</td>
<td>GEN-HP</td>
<td>26.2 rad/s</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) ILP</td>
<td></td>
<td>(b) IHP</td>
<td></td>
</tr>
<tr>
<td>FO-ILP</td>
<td>628.3 rad/s</td>
<td>FO-IHP</td>
<td>2.28 krad/s</td>
</tr>
<tr>
<td>PL-ILP</td>
<td>3.297 krad/s</td>
<td>PL-IHP</td>
<td>344.9 rad/s</td>
</tr>
<tr>
<td>GEN-ILP</td>
<td>37.88 krad/s</td>
<td>GEN-IHP</td>
<td>30.1 rad/s</td>
</tr>
</tbody>
</table>

7.3.1 Time-domain performance

In the frequency domain, a graph reveals the distribution of a signal across various frequency bands within a given frequency range. On the other hand, in the time domain, a graph illustrates how a signal changes over time. Frequency-domain analysis involves units like Hertz or rad/s, while time-domain analysis employs time units such as seconds, milliseconds etc.

This subsection includes an indicative examination of the time-domain behaviour for the case of generalized low-pass and inverse high-pass filters. The evaluation requires stimulation with a sinusoidal input at a frequency equal to the half-power frequency for each case. For the low-pass filter, this frequency is $27.96 \text{ krad/s}$, and for the inverse high-pass filter, it is $30.096 \text{ rad/s}$. The amplitude of the signal is set to 40mV. The obtained waveforms are depicted in Fig.7.4a and 7.4b.

With data derived from these graphs and using the equations given below:

$$Gain_{dB} = 20 \cdot \log\left(\frac{V_{out}}{V_{in}}\right) \quad \text{and} \quad phase = 360^\circ \cdot \frac{d}{D}$$

where D is the period of the signal and d is the time difference between the input and output.

For the low-pass filter, we result to the values of $Gain_{LP,GEN} = -3.5 \text{ dB}$ and $Gain_{ideal,LP,GEN} = -3.6 \text{ dB}$. For the phase difference, we get $phase_{LP,GEN} = 32.16^\circ$ and $phase_{ideal,LP,GEN} = 32.3^\circ$. For the Inverse high-pass filter and while using the same formulas as before, we get $Gain_{IHP,GEN} = 3.6 \text{ dB}$ and $Gain_{ideal,IHP,GEN} = 3.4 \text{ dB}$. As for the phase difference we result to $phase_{IHP,GEN} = 7.67^\circ$ with the theoretical value being $phase_{ideal,IHP,GEN} = 6.15^\circ$. 
7.3 Simulation results

![Graphs showing generalized low-pass and inverse high-pass input and output waveforms](image)

**Fig. 7.4** Time-domain performance of the generalized low-pass (a) and generalized inverse high-pass (b) input and output waveforms for a sinusoidal input at their cutoff frequencies and 40mV amplitude

### 7.3.2 Distortion

Input-referred noise is the noise voltage that, when applied to the input of the noiseless circuit, generates the same output noise as the actual circuit does. This model is a method of simplifying the noise analysis in electronic circuits, defining a noise floor for the signal. The Total Input Referred Noise was measured in Cadence indicatevely for the cases of low-pass fractional-order, power-law and generalized filters. The rms values are $38.4\mu V$, $34.0\mu V$ and $39.7\mu V$ accordingly.

The evaluation of the Total Harmonic Distortion (THD) is a way of determining how much a signal is distorted by noise. A signal’s harmonics are multiples of its main frequency. The objective is to examine how much additional energy is present at harmonic frequencies in comparison to the original signal. THD is expressed as a percentage, corresponding to the proportion of additional harmonic energy to the total signal strength. Thus, a lower THD % indicates a less distorted signal. Acceptable THD values in this thesis are those that are lower than 1%.

Cadence software’s ‘thd()’ function calculates total harmonic distortion as a percentage. Signal amplitudes are altered between 10mV and 100mV to evaluate the influence on THD levels. Using this process, the amplitude associated with a THD of 1% can be determined and stored for further use in the Dynamic Range equation. This identified amplitude serves as the upper limit for the signal range, ensuring the highest permissible value while maintaining signal integrity.

The Tables 7.1a, 7.1b and 7.1c demonstrate the THD data for each case and an additional value with the amplitude of 1%. **Fig. 7.5** depicts the THD graph for the low-pass filter of every category.
Layout and Post-Layout

<table>
<thead>
<tr>
<th>Fractional Order</th>
<th>Power Law</th>
<th>Generalized Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude (mV)</td>
<td>THD (%)</td>
<td>Amplitude (mV)</td>
</tr>
<tr>
<td>10</td>
<td>5.375m</td>
<td>10</td>
</tr>
<tr>
<td>20</td>
<td>5.375m</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>68.74m</td>
<td>30</td>
</tr>
<tr>
<td>40</td>
<td>212.6m</td>
<td>40</td>
</tr>
<tr>
<td>50</td>
<td>486.7m</td>
<td>50</td>
</tr>
<tr>
<td>60</td>
<td>921.5m</td>
<td>60</td>
</tr>
<tr>
<td>63</td>
<td>1.086</td>
<td>70</td>
</tr>
<tr>
<td>70</td>
<td>1.532</td>
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<td>90</td>
<td>3.264</td>
<td>95</td>
</tr>
<tr>
<td>100</td>
<td>4.345</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 7.2 Total Harmonic Distortion (THD) data for low-pass, fractional-order (a), power-law (b) and generalized-order (c) filters with varying amplitude.

Dynamic Range, indicated as DR, is the proportion between the highest and lowest values that a particular quantity can demonstrate. This concept is widely used in the context of signals. Dynamic range is measured by computing the ratio of the highest and lowest signal values – namely the upper limit and the noise floor, and expressing it in base-10 logarithms (decibels), as follows in Eq. (7.1). Notingly, for the case of power-law filter, according to Fig. 7.5 THD consistently displays the lowest values across the varying amplitudes, so a higher Dynamic Range value is expected.

\[
DR = 20 \log \left( \frac{\text{input rms at 1\%THD ampl.}}{\text{rms value of Total Input Referred Noise}} \right)
\]

\[
= 20 \cdot \log \left( \frac{\text{input rms at 1\%THD ampl.}}{\text{rms value of Total Input Referred Noise}} \right) \tag{7.1}
\]

To calculate the rms value at 1% THD, a multiplication of the values of the selected amplitudes from Tables 7.1a, 7.1b and 7.1c by the factor of \( \frac{1}{\sqrt{2}} \) is needed.

The resulting Dynamic Range of the Low-pass Filters are the following: \( DR_{FO} = 61.3 \) dB, \( DR_{PL} = 67 \) dB and \( DR_{GEN} = 62.51 \) dB. The value for the Power-Law Filter is indeed the highest among the three cases.
7.3 Simulation results

![Graph showing THD vs. Amplitude](image)

**Fig. 7.5** Linear performance of all low-pass filters for a 62.8 rad/s stimulus of variable amplitude.

### 7.3.3 Monte Carlo analysis

The sensitivity performance of the generalized low-pass and inverse high-pass filters is evaluated by employing the Monte Carlo analysis. The derived histograms of the cutoff frequency for both filters are presented in Figs. 7.6a and 7.6b, where the mean value and the standard deviation of the considered frequency characteristic are 28.07 krad/s, 0.556 krad/s, and 1.676 krad/s, 0.067 krad/s, respectively, confirming the robustness of the proposed implementation. The obtained statistical plots confirm that the proposed implementation offer reasonable sensitivity characteristics.

![Histograms of cutoff frequency](image)

**Fig. 7.6** Monte-Carlo analysis statistical histograms of the cutoff frequency in the case of generalized (a) low-pass and (b) inverse high-pass filters.
7.3.4 Power Dissipation

In analog circuits, power dissipation refers to the quantity of electrical power consumed or lost as heat by the circuit’s components. This phenomena arises from the non-ideal properties of real-world components such as resistors, transistors, and other electronic elements. Excessive heat can compromise component performance and reliability, making power dissipation an important factor in analog circuits.

Indicatively, the cases of the lowpass and highpass filter were tested in respect to their power dissipation.

The DC power dissipation is derived from $P = (V_+ - V_-)I_{dis}$, where $I_{dis}$ is the current dissipated from the positive $V_+$ to the negative $V_-$ voltage supply. The currents are 1.349 $\mu$A, 2.889 $\mu$A and 2.431 $\mu$A for the cases of fractional-order, power-law and generalized low-pass filter. The corresponding currents for the inverse high-pass cases are 2.1384 $\mu$A, 941.1 nA and 2.276 $\mu$A respectively. Concluding, the power dissipation values for the low-pass filters are 2 $\mu$W, 4.33 $\mu$W, and 3.65 $\mu$W, while for the inverse filters they are 3.21 $\mu$W, 1.41 $\mu$W, and 3.41 $\mu$W.
Chapter 8

Conclusions and future work

8.1 Conclusions

The primary goal of this Thesis is to design and implement non-integer order filters derived from a first-order mother transfer function. More concretely, the cases of fractional-order, power-law and double-order low-pass, high-pass, and band-pass filters, as well as their Inverse equivalents, are examined. To effectively implement the filters under consideration, the derived transfer functions must be approximated. Since standard approaches cannot be used to approximate generalized transfer functions, a curve fitting method based on the fitfrd command in MATLAB is utilised. The modification of already existing codes in order to approximate any filter case and exponent combination is one of the main tasks and, also, the contribution of this work. The resulting integer-order transfer functions can be implemented employing conventional filter design topologies. This work utilizes the Inverse-Follow-the-Leader-Feedback multiple-feedback filter topology and OTAs as the active element for its realization, taking advantage of the adjustability of the transconductance by fine-tuning the DC bias currents. This structure offers flexibility by allowing the transfer functions of the filters to be implemented from the same active core, with adjustments to bias currents. Post-layout simulation results validate the reliability of this approach. In summary, the proposed concept provides advantages such as resistorless implementations, satisfactory accuracy, utilization of only grounded capacitors, accurate approximation of any order case transfer function and a universally adaptable electronic structure.
Conclusions and future work

8.2 Proposal for future work

The system’s efficiency, established in previous chapters, makes it a valuable tool for various applications. Some suggestions for future study endeavours include:

- Fabrication of the circuit in order to experimentally verify its behavior.
- Research for different structures to reduce the total number of active elements and transistors.
- Performance evaluation of other types of filters such as band-stop, Elliptic and Bessel, among others, to assess their effectiveness and applicability.
- Search and comparison of different approximation techniques.
- Employment of other types of active elements and structures for the implementation.
- Verification of the impact of order and the exponents’ combination on frequency characteristics.
- Derivation of the optimal combination of orders for minimizing the noise using genetic algorithms or Machine Learning techniques.
- Control applications, such as mechatronics and motion control systems, focusing on utilizing the entire transfer function’s magnitude response for approximation.
- Realization of the filters using Field Programmable Analog Arrays (FPAAs) and comparison with the proposed work.
- Identifying and characterizing materials with fractional impedance behavior that are also environmentally friendly, i.e. natural materials or synthetic compounds that mimic the properties of biological tissues. This exploration could lead to the development of sustainable electronic devices with applications in healthcare, consumer electronics, and renewable energy.
References


References


References


References


Appendix A

MATLAB codes

A.1 MATLAB code for the curve fitting approximating technique for non-integer order filters

The provided MATLAB code extracts the integer order approximate transfer function corresponding to the user-defined non-integer order function, along with essential characteristics required for subsequent implementation, such as transconductance (gm), basic bias currents, and scaling factors.

close all;
clear all;
clearvars;
clc;
format shortEng;

%% user data
alpha = 0.3;
beta = 0;
gamma = 1;
Go=1; % gain factor
order = 4; % approximation order for fitting
wp=1E+3;
cap = 100E-12; % capacitor in pico-Farads
ib = 10E-09; % bias current 10nA
n = 1.5;  % slope factor of a MOS transistor in sub-threshold region
vt = 26E-03;  % thermal voltage = 26 mV at 27 Â°C

%%% internal calculations
tau = 1 / wp;
wmin = wp * 1E-02;  % minimum frequency wmin
wmax = wp * 1E+02;  % maximum frequency wmax

% Create a logarithmically spaced frequency vector between wmin and wmax
w = logspace (log10(wmin), log10(wmax), 1000);

%%% ideal transfer function
G_ID = ((tau*s)^beta/(1+((tau*s)^alpha)))^gamma;

%%% 2 step procedures from ideal transfer function
% Create an intermediate FOTF using fotf function
H_ID = fotf([tau.^alpha 1], [alpha 0], [tau.^beta 0], [beta 0], 0)

% Calculate the frequency response of the intermediate FOTF
H_ID_FREQ_RESP = freqresp(H_ID, w);
% Raise the intermediate response to the power of gamma to get the ideal response and multiply by the Gain factor
H_ID_FREQ_RESP = Go*(H_ID_FREQ_RESP.^gamma);
% Create a frequency response data (FRD) object from the ideal response
H_ID_RESP_DATA = frd(H_ID_FREQ_RESP, w);
% Fit the FRD data to a transfer function of the specified order
H_ID_FREQ_ORDER = fitfrd(H_ID_RESP_DATA, order);
% Compute the minimal realization of 'H_FIT_FRD'
H_FIT_FRD = minreal(tf(H_ID_FREQ_ORDER))  % approx tf
A.1 MATLAB code for the curve fitting approximating technique for non-integer order filters

```matlab
[num, den] = tfdata(H_FIT_FRD, 'v'); % extracts numerator and denominator

for i = 1: (order)
    time_const(i) = den(1,i) / den(1,i + 1);
    gm(i) = cap / time_const(i);
    Io(i) = (9*n*vt*gm(i))/5; % Theoretical bias currents of OTAs
end

[gm] = gm'; % transconuctances
[Io] = Io';

for z = 0: order
    G(z+1) = num(1,order+1-z)/den(1,order+1-z);
end

[G] = G'; % scaling factors
Io = vertcat(ib, Io); % Io(1)=ib=10nA

%% Bode plots
% Compute the Bode plot (gain and phase) of the ideal transfer function
[gain_id, phase_id, w] = bode(H_ID, w);
w = squeeze(w);
gain_id = squeeze(gain_id);
gain_id = Go*(gain_id.^gamma);
gain_id_dB = mag2db(gain_id); % Convert gain to decibels
phase_id = squeeze(phase_id);
phase_id = gamma.*phase_id;

% Compute the Bode plot of the fitted transfer function 'H_FIT_FRD'
[gain_fit_frd, phase_fit_frd, w] = bode(H_FIT_FRD, w);
gain_fit_frd = squeeze(gain_fit_frd);
gain_fit_frd_dB = mag2db(gain_fit_frd);
phase_fit_frd = squeeze(phase_fit_frd);
```

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%% Error calculation
err_gain_fit_frd = abs((gain_fit_frd_dB - gain_id_dB));
err_phase_fit_frd = abs((phase_fit_frd - phase_id));

%% Magnitude plots
figure;
semilogx(w, gain_id_dB, 'k-', 'MarkerSize', 18, 'LineWidth', 2);
hold on;
semilogx(w, gain_fit_frd_dB, 'r', 'MarkerSize', 18, 'LineWidth', 2);

% Plot title labels etc.
grid on;
title('\textit{Gain response}', 'Fontsize', 14);
set(gca, 'FontSize', 16);
xlabel('Frequency (rad/s)', 'FontSize', 16);
ylabel('Gain (dB)', 'FontSize', 16);
legend({'\textit{ideal}', '\textit{fitfrd}'}, 'Location', 'southwest');

%% Phase plots
figure;
semilogx(w, phase_id, 'k-', 'MarkerSize', 18, 'LineWidth', 2);
hold on;
semilogx(w, phase_fit_frd, 'r', 'MarkerSize', 18, 'LineWidth', 2);

% Plot title labels etc.
grid on;
title('\textit{Phase response}', 'Fontsize', 14);
set(gca, 'FontSize', 16);
xlabel('Frequency (rad/s)', 'FontSize', 16);
ylabel('Phase (deg)', 'FontSize', 16);
legend({'\textit{ideal}', '\textit{fitfrd}'}, 'Location', 'southwest');
A.1 MATLAB code for the curve fitting approximating technique for non-integer order filters

%% Magnitude error plots
% Determination of the plot
figure;
semilogx(w,err_gain_fit_frd,'b','MarkerSize',18,'LineWidth',2);
hold on;

% Plot title labels etc.
grid on;
title('\fontsize{14} Gain error','Fontsize',14);
set(gca,'fontsize',16);
xlabel('Frequency (rad/s)','Fontsize',16);
ylabel('Error (dB)','Fontsize',16);
legend({'fitfrd'},'Location','northwest');

%% Phase error plots
% Determination of the plot
figure;
semilogx(w,err_phase_fit_frd,'b','MarkerSize',18,'LineWidth',2);
hold on;

% Plot title labels etc.
grid on;
title('\fontsize{14} Phase error','Fontsize',14);
set(gca,'fontsize',16);
xlabel('Frequency (rad/s)','Fontsize',16);
ylabel('Error (deg)','Fontsize',16);
legend({'fitfrd'},'Location','north');
MATLAB codes