Professor Emeritus Nikolaos I. Ioakimidis, former professor at the

DIVISION OF APPLIED MATHEMATICS AND MECHANICS
DEPARTMENT OF ENGINEERING SCIENCES
SCHOOL OF ENGINEERING, UNIVERSITY OF PATRAS

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Quantifier elimination and quantifier-free formulae for universally–existentially (AE) quantified formulae in Ben-Haim's info-gap model of uncertainty

Nikolaos I. Ioakimidis

School of Engineering, University of Patras, GR-265 04 Patras, Greece
e-mail: n.ioakimidis@upatras.gr

Abstract The method of quantifier elimination with implementations in some computer algebra systems already proved useful for the computation of both the robustness and the opportuneness (or opportunity) functions in Ben-Haim's info-gap (or information-gap) model of uncertainty. As is well known, this model constitutes an interesting and practical tool in decision theory. Moreover, quantifier elimination concerning the robustness/opportuneness functions can be performed to the related universally/existentially quantified formulae. Here we proceed to the consideration of the additional mixed (AE) case, where both the universal and the existential quantifiers are present in the quantified formula related to Ben-Haim's info-gap model of uncertainty. In this mixed (AE) case, evidently now with more than one uncertain variable, the universal quantifier concerns one (or more than one) uncertain variable and similar is the case with the existential quantifier. After performing quantifier elimination to this quantified formula (here by using the computer algebra system *Mathematica*), we derive the related QFF (quantifier-free formula) that concerns the horizon of uncertainty. The case of more than one horizon of uncertainty can also be similarly studied. In this way, an expression for the horizon of uncertainty in a logical form with the appropriate inequalities is derived. From this form it is observed that additional immunity functions (beyond the classical robustness and opportuneness functions) appear in the mixed universal–existential (AE) case. The present approach is applied to four uncertainty problems which are based on info-gap models and concern (i) the area of a rectangle, (ii) the buckling load of a fixed–free column, (iii) the volume of a rectangular cuboid and (iv) the reactions at the ends of a fixed beam loaded by a concentrated load.

Keywords Uncertainty · Info-gap · Information-gap · IGDT · Non-probabilistic methods · Robust reliability · Robustness · Opportuneness · Universal quantifier · Existential quantifier · Quantified formulae · Quantifier elimination · Quantifier-free formulae · Rectangles · Area · Rectangular cuboids · Volume · Columns · Buckling load · Beams · Reactions · *Mathematica*

1. Introduction

As is well known, quite frequently, uncertainty conditions are present in problems that have to be solved in engineering and in many other scientific disciplines. These uncertainties concern one or more than one parameter in the problem under consideration. For the efficient solution of such a problem one can employ either (i) probabilistic methods or (ii) non-probabilistic methods.

Probabilistic methods are very accurate, but, unfortunately, they have the disadvantage that the related probability density functions for the quantities of interest should be known or computed in advance. Therefore, probabilistic methods are complicated in their practical use or, simply, they cannot be used because of the lack of the necessary data, which are required for the computation of the related probability density functions.

On the contrary, non-probabilistic methods do not use probability density functions and, hence, they are much simpler in their practical use in problems under uncertainty conditions. Therefore, they are often preferred over probabilistic methods. Two very well-known and quite frequently used non-probabilistic methods are (i) interval analysis and (ii) convex models and, especially, the popular ellipsoidal convex model and its various generalizations.

Interval analysis is a very old approach, but it has become very popular in its modern form mainly after the publication of the related results by Moore and his collaborators since 1959. The classical books on modern interval analysis are those by Moore [1] published in 1966 and by Moore, Kearfott and Cloud [2] published in 2009.

On the other hand, with respect to convex models reference should be made to the classical and pioneering book by Ben-Haim and Elishakoff [3] published in 1990. Several applied mechanics problems under uncertainty conditions are studied in this book by using the method of convex models. Among many convex models the ellipsoidal model is particularly popular and practically useful. The ellipsoidal model has been used by Schweppe [4, 5] since 1968 in problems concerning dynamic systems. Several generalizations of the ellipsoidal model and also a series of additional convex models are frequently used in problems under uncertainty conditions. Of course, this class of problems includes applied mechanics problems, which are of particular interest to this author.

A very interesting extension of the method of convex models for the study of problems under uncertainty conditions is the IGDT (info-gap or information-gap decision theory). This theory has been proposed by Ben-Haim. Ben-Haim's IGDT constitutes a popular, efficient and very useful methodology in problems under uncertainty conditions and it has been applied to a very large number of problems under uncertainty conditions appearing in several scientific disciplines. The IGDT was originally proposed by Ben-Haim in his interesting book concerning robust reliability in the mechanical sciences [6] and published in 1996, but its present name, IGDT (info-gap or information-gap decision theory), was proposed (also by Ben-Haim) later. This new and very interesting method is the method that will be exclusively used here. The classical book on the IGDT is still the book by Ben-Haim published in 2001 [7] with its second edition published in 2006 [8]. A related book by Ben-Haim is devoted exclusively to info-gap economics [9]. The IGDT is also described in detail in several publications by Ben-Haim and his collaborators. These publications include the papers [10] by Ben-Haim, [11] by Ben-Haim, Cogan and Sanseigne, [12] by Ben-Haim and Laufer and [13–19] by Ben-Haim.

Here it is important to mention that the three main elements (or components) of the IGDT are

1. the model of the system under consideration,
2. the assumed info-gap model of uncertainty and
3. the performance requirement (or, frequently, performance requirements) of the system model.

Here we will also make some additional assumptions, more explicitly, some positivity assumptions that correspond to the physical properties of the uncertain quantities, which are frequently positive and, additionally, they constitute a significant help to quantifier elimination. Moreover, the related decision functions, which are sometimes called immunity functions, are

1. the robustness function (or simply robustness) and
2. the opportuneness (or opportunity) function (or simply opportuneness).

Nevertheless, it is clear that the robustness function is much more useful than the opportuneness (or opportunity) function in practical problems of course always under uncertainty conditions. Several additional related functions "lying" between the robustness and the opportuneness functions appear in the present four applications presented in Section 3, Section 4, Section 5 and Section 6 below.

An very large number of publications are devoted to various interesting applications of Ben-Haim's IGDT (info-gap or information-gap decision theory). These publications concern many problems under uncertainty conditions that appear in several different disciplines.

Here we make reference to the papers (in chronological order) by Hipel and Ben-Haim [20], Kanno and Takewaki [21], Takewaki and Ben-Haim [22], Regev, Shtub and BenHaim [23], Duncan, Bras and Paredis [24], Ben-Haim, Dacso, Carrasco and Rajan [25], Ben-Haim and Cogan [26], Ben-Haim [27], Ben-Haim and Hemez [28], Wang, F., Zhang, J., Wang, X., Wang, C. and Liu, Z. [29], Piegat and Tomaszewska [30, 31], Matrosov, Woods and Harou [32], Hayes, Barry, Hosack and Peters [33], Maugan, Cogan, Foltête, Buffe and Kerschen [34], Ben-Haim, Irias and McMullin [35], Wu, D., Gao, W., Li, G., Tangaramvong and Tin-Loi [36], Roach, Kapelan and Ledbetter [37], Tomaszewska and Piegat [38], Kanno, Fujita and Ben-Haim [39], Hot, Weisser and Cogan [40], Ben-Haim and Cogan [41], Sun, B., Li, S., Xie, J. and Sun, X. [42], Jaboviste, Sadoulet-Reboul, Peyret, Arnould, Collard and Chevallier [43], Hemez and Van Buren [44], Jabari, Mohammadi-ivatloo, Ghaebi and Bannae-Sharifian [45], Rezaei, Ahmadi, Nezhad and Khazali [46], Dai, X., Wang, Y., Yang, S. and Zhang, K. [47], Kuczkowiak, Cogan, Ouisse, Foltête and Corus [48], Nojavan and Jermsittiparser [49], Zhao, E. and Wu, C. [50], Housh and Aharon [51], Li, X, Li, X., Zhou, Z., Su, Y. and Cao, W. [52], Ben-Haim [53] and Liu, Y., Wang, P., Thomas, M. L., Zheng, D. and McKirdy, S. J. [54].

Moreover, a very large number of additional interesting publications (appropriately classified) by employing Ben-Haim's IGDT can be found through the web page https://info-gap.technion.ac.il of the Technion, Israel Institute of Technology. This web page is devoted to Ben-Haim's IGDT. On the other hand, the paper by Kanno [55] refers to the worst scenario method, which is somewhat relevant to the IGDT. Finally, the IGDT-related approach used by Au, Cheng, Tham and Zeng [56] and by Cheng, Au, Tham and Zeng [57] is also of some interest.

Here we proceed in our recent research related to Ben-Haim's IGDT (info-gap or information-gap decision theory) by using again the computer algebra system *Mathematica* as well as the computational method of quantifier elimination implemented in *Mathematica* in order to perform symbolic computations. Of course, our present results are again related to the IGDT. It is completely understood that for symbolic computations (as is here again the case) the use of a computer algebra system is particularly convenient. Symbolic computations proved very useful in a variety of problems of applied mechanics and of many other scientific disciplines since about the sixties.

The most well-known computer algebra systems are *ALTRAN* (1965), *REDUCE* (1966), *Macsyma* (1968) (since 1982 also in its free version *Maxima*), *muMath* (1978), *Maple* (1982), *Derive* (1988) and *Mathematica* (1988). An interesting review paper concerning the use of symbolic computations in applied and structural mechanics was prepared by Pavlović [58] and was published in 2003. On the other hand, in his own research since 1989, the author used four computer algebra systems: (i) *Derive*, (ii) *REDUCE*, (iii) *Maple* and (iv) *Mathematica* in several problems of applied mechanics where there exists an intensive necessity for symbolic computations.

As has been already mentioned, the present results, which exclusively concern Ben-Haim's IGDT [8, 10] here by using the computational method of quantifier elimination, were again obtained with the use of *Mathematica*. In fact, *Mathematica* has again been selected by the author because

1. in its kernel it includes an efficient implementation of several quantifier elimination algorithms prepared by Strzeboński and, additionally,
2. it offers a very powerful and friendly to the user computational environment.

Quantifier elimination for real variables implemented in some computer algebra systems that will be mentioned below, is a very useful computational tool strongly related to computer algebra. We can perform quantifier elimination to quantified formulae, that is to formulae that include either

1. the universal quantifier $\forall$ (for all) or
2. the existential quantifier $\exists$ (exists) or
3. simultaneously both of these quantifiers.

In this way, we obtain formulae that are free from these quantifiers and the related quantified variables. Such formulae are usually called QFFs, that is quantifier-free formulae. On the contrary, evidently, the free variables are not eliminated during quantifier elimination and they remain present in the resulting QFFs. An interesting extensive bibliography concerning various applications of quantifier elimination for real variables was prepared by Ratschan [59] in 2012.

Among many algorithms concerning quantifier elimination for real variables two are those that are mostly used in practice. The first such quantifier elimination algorithm is CAD (cylindrical algebraic decomposition). This extremely useful and popular algorithm was devised by Collins in 1973 with its official publication appeared in 1975 [60]. The second such quantifier elimination algorithm is virtual substitution. This also interesting and useful algorithm was devised by Weispfenning in 1988; see, e.g., Refs. [61, 62] as well as the recent invited paper by Sturm [63], which was prepared on the occasion of the completion of thirty years of virtual substitution. At this point it should be mentioned that CAD is a general-purpose algorithm for quantifier elimination whereas, on the contrary, virtual substitution is mainly applicable to quantified formulae that include only linear, quadratic and cubic polynomial constraints. For linear and quadratic polynomial constraints virtual substitution is more efficient than CAD and, therefore, it is preferable to CAD. The classical book on quantifier elimination and CAD is still the well-known book edited by Caviness and Johnson [64]. This book was published in 1998. A very large number of improvements of CAD are also available; see, e.g., the paper by Strzeboński [65] concerning the use of local projections, the recent paper by England, Bradford and Davenport [66] concerning CAD with equational constraints and the very recent paper by Li, H., Xia, B., Zhang, H. and Zheng, T. [67] concerning the selection of the variable ordering for CAD.

The best implementation of CAD (cylindrical algebraic decomposition) seems to be that made by Strzeboński in *Mathematica* [68] and this is the implementation that will be exclusively used here. But on the other hand, unfortunately, from the negative point of view it should be mentioned that quantifier elimination for real variables has a doubly-exponential computational complexity. This important property of quantifier elimination was proved by Davenport and Heintz [69], it is well known and, naturally, applicable to CAD as well. This negative result constitutes a serious disadvantage of the method of quantifier elimination and, therefore, a significant obstacle to its wide application especially to quantified formulae with a large total number of variables (both free and quantified variables) and/or high-degree polynomials. On the other hand, the new DEWCAD project, which was recently very briefly described by Bradford, Davenport, England, Sadeghimanesh and Uncu [70] and aims at pushing back the doubly-exponential wall (DEW) of CAD, is also of particular importance as far as CAD is concerned.

Five major implementations of quantifier elimination algorithms for real variables in computer algebra systems are known to this author. These implementations (mainly based on CAD and/or on virtual substitution) are

1. the classical and famous package QEPCAD (now QEPCAD B [71]) of the *SACLIB* library; this is the first computer package that performs quantifier elimination,
2. the package REDLOG [72] of the computer algebra system *REDUCE*,
3. the package SyNRAC [73] of the computer algebra system *Maple*,
4. the recent and based on a very interesting concept (the concept of being poly-algorithmic) package QuantifierElimination [74] also of *Maple* and
5. the implementation of quantifier elimination in *Mathematica* [68] described in Refs. [75, 76]; this implementation is very efficient inside the powerful, user-friendly and modern computational environment offered by *Mathematica* [68] and it is the implementation that will be exclusively used in the present technical report.

At this point it should also be mentioned that the `SemiAlgebraicSetTools` subpackage of the `RegularChains` package of *Maple* includes among its recent improvements for *Maple* 2020 the new command `QuantifierElimination`, which performs quantifier elimination for real variables.

Since 1994 the author following many researchers in several research fields has been interested in the application of the method of quantifier elimination for real variables [64] to some problems of applied mechanics; see, e.g., Ref. [77] published in 1995 and Ref. [78] published in 1999. More recent results by the author, but also based on quantifier elimination [64] and generally concerning problems of applied mechanics, can be found in several more recent publications that include Refs. [79–92]. On the other hand, Charalampakis and Chatzigiannelis directly applied the Collins' CAD algorithm to the problem of minimum weight design of trusses in structural mechanics [93].

Of course, with respect to the present results much more interesting seems to be the application of quantifier elimination for real variables to problems concerning the computation of intervals for the uncertain quantities. The author was interested in this type of applications during the last three years. The related results by the author include the nine recent technical reports [80–88], where the author combined interval analysis with quantifier elimination (by using (i) its implementation in *Mathematica* [68] in the first seven of these technical reports [80–86] as well as in the last technical report [88] and (ii) REDLOG [72] in *REDUCE* in the eighth technical report [87]) in several problems. Almost all of these problems concern applied mechanics with the exception of the problems studied in the technical report [81]; these problems concern intervals of the real roots of the quadratic equation. On the other hand, quite recently, the author applied quantifier elimination (i) to the ellipsoidal model under uncertainty conditions [89, 90] with the emphasis put again on problems of applied mechanics as well as (ii) to Ben-Haim's IGDT [91, 92]. Here by generalizing the results of Refs. [91, 92] (mainly the results of Ref. [91]) we intend to consider another aspect of the IGDT of course continuously using the computational method of quantifier elimination and *Mathematica*.

The present results constitute a simple but reasonable generalization of the very recent results of Ref. [91] (see also Ref. [92]) concerning Ben-Haim's IGDT by using the method of quantifier elimination. More explicitly, here we also study the mixed case of universally–existentially quantified formulae with an emphasis put on the resulting new immunity functions in the derived QFFs (quantifier-free formulae) beyond the two so well known robustness and opportuneness functions. Of course, it is understood that in the mixed case studied here the number of quantified variables in the quantified formula should be at least two. This will permit that both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) can be simultaneously present in the quantified formulae. It is also completely understood that although the present applications are either geometric (those in Section 3 and in Section 5) or engineering concerning applied mechanics, more explicitly mechanics of materials or strength of materials (those in Section 4 and in Section 6), nevertheless, the methodology that is followed here is quite general. Hence, it is applicable to problems in any scientific discipline that can be studied by the IGDT provided, of course, that such a problem falls inside the undoubtedly practically restricted computational possibilities of quantifier elimination.

The present technical report is organized in seven sections as follows:

1. In Section 1 (the present section), we briefly present the introductory material to the adopted approach, which combines the use of Ben-Haim's IGDT (info-gap or information-gap decision theory) [7, 8, 10] with the computational method of quantifier elimination [64] for problems related to uncertainty conditions.
2. In Section 2, we briefly introduce the extremely well-known Ben-Haim's robustness and opportuneness (or opportunity) functions by displaying the related quantified formulae. These functions are the two classical immunity functions of the IGDT. Additionally, we display the (mixed) universally−existentially (AE) quantified formulae concerning systems of interval linear equations, which inspired the present research, also providing a very small part of the related extensive literature. Finally, in the same section, Section 2, we also display the general form of the corresponding (mixed) universally−existentially (AE) quantified formula to be employed in the present applications of Ben-Haim's IGDT.

Next, the subsequent four sections concern applications of the IGDT. In these applications, we use

1. the universally quantified formulae related to the corresponding robustness functions,
2. the existentially quantified formulae related to the corresponding opportuneness functions
3. the present (mixed) universally−existentially (AE) quantified formulae related to the corresponding new immunity functions concerning the present (mixed) AE-case.

More explicitly, we study four problems of course under uncertainty conditions concerning

1. the area of a rectangle (in Section 3),
2. the buckling load of a fixed−free column (in Section 4),
3. the volume of a rectangular cuboid (in Section 5). and
4. the reactions at the ends of a fixed ordinary beam loaded by a concentrated load (in Section 6).
5. Finally, in Section 7, we state the basic conclusions that are drawn from the present research results and we also make a brief related discussion. In this discussion, emphasis is put on the present mixed AE-case in the quantified formulae for Ben-Haim's info-gap model and on the new immunity functions that result in this particular case.

The present results concern the mixed AE-case where there exists at least one value (or one set of values) of the existentially quantified variable(s) for which the performance requirement(s) $\mathscr{C}$ in the uncertainty model hold true for all values (or for all sets of values) of the universally quantified variable(s). Of course, this happens under the validity of the assumed info-gap model $\mathscr{U}$ and here we have selected fractional-error info-gap models in all applications below. On the other hand, the derivation of the related conditions of existence may be of practical interest in design problems.

2. Universally, existentially and universally−existentially quantified formulae in the IGDT

2.1. The robustness and opportuneness functions and the related quantified formulae

As is well known, the robustness function (or, simply, robustness) $\widehat{\alpha}$ and the opportuneness (or opportunity) function (or, simply, opportuneness) $\widehat{\beta}$ are the the two main functions in Ben-Haim's IGDT (info-gap or information-gap decision theory); see, e.g., the classical book on the IGDT by Ben-Haim [8, Chap. 3, pp. 37–114]. This chapter is devoted to these two fundamental functions. The same functions of the IGDT are frequently called immunity functions; see, e.g., [8, pp. 38−40]. These functions, $\widehat{\alpha}$ and $\widehat{\beta}$, are functions of the parameters in the problem under consideration. These parameters constitute a decision vector q of parameters, that is $\widehat{\alpha} = \widehat{\alpha}(q)$ and $\widehat{\beta} = \widehat{\beta}(q)$ [8, p. 38].

At first, as far as the robustness function $\widehat{\alpha}$ is concerned, this function is defined as the maximum value of the uncertainty parameter (or horizon of uncertainty) h for which the minimal requirements (the performance requirements) $\mathscr{C}$ of the assumed uncertainty model are always satisfied [8, p. 38, Eq. (3.1)], i.e. they are satisfied for all sets of values of the parameters q belonging to the assumed info-gap model. This equation (here in a slightly different notation) has the form [8, p. 38, Eq. (3.1)]

$$\widehat{\alpha} = \widehat{\alpha}(q) := \max\{h \mid \text{minimal requirements are always satisfied}\} \quad \text{(robustness)}. \tag{1}$$

The robustness function $\widehat{\alpha}$ expresses the robustness of the uncertainty model, i.e. the degree of resistance to uncertainty and the immunity against failure; so a large value of $\widehat{\alpha}$ is desirable [8, p. 38].

On the other hand, as far as the opportuneness function $\widehat{\beta}$ is concerned, this second but also very important function in the IGDT is defined as the minimum value of the same uncertainty parameter (or horizon of uncertainty) h for which a sweeping success is possible, i.e. the minimal requirements (the performance requirements) $\mathscr{C}$ of the assumed uncertainty model can be satisfied [8, p. 38, Eq. (3.2)]. Here this means that these requirements $\mathscr{C}$ are satisfied for at least one set of values of the parameters q that belong to the assumed info-gap model $\mathscr{U}$. This equation (here again in a slightly different notation) has the form [8, p. 38, Eq. (3.2)]

$$\widehat{\beta} = \widehat{\beta}(q) := \min\{h \mid \text{sweeping success is possible}\} \quad \text{(opportuneness)}. \tag{2}$$

The opportuneness (or opportunity) function $\widehat{\beta}$ expresses the minimum level of uncertainty h which must be tolerated in order to enable the possibility of a sweeping success; so a small value of $\widehat{\beta}$ is desirable [8, p. 38].

Here we will employ the two related quantified formulae with the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) with respect to the robustness and opportuneness functions, respectively. We also take into account the three main elements (components) in an IGDT problem and we denote the system model by the symbol $\mathscr{M}$, the adopted info-gap model by the symbol $\mathscr{U}$ and the performance requirement(s) by the symbol $\mathscr{C}$. We also take into account our assumptions (mainly positivity assumptions) $\mathscr{A}$. Then the aforementioned quantified formulae take the forms

$$\begin{aligned} \forall q\ \ & \text{in the info-gap model } \mathscr{U} \text{ such that the system model } \mathscr{M} \text{ and the assumptions } \mathscr{A} \\ & \text{hold true the performance requirement(s) } \mathscr{C} \text{ also holds (hold) true,} \end{aligned} \tag{3}$$

which is a universally quantified formula and

$$\begin{aligned} \exists q\ \ & \text{in the info-gap model } \mathscr{U} \text{ such that if the system model } \mathscr{M} \text{ and the assumptions } \mathscr{A} \\ & \text{hold true, the performance requirement(s) } \mathscr{C} \text{ also holds (hold) true,} \end{aligned} \tag{4}$$

which is an existentially quantified formula. These two formulae correspond to Eqs. (1) and (2), respectively.

Now we can perform quantifier elimination to the two quantified formulae (3) and (4). Then we obtain the related QFFs (quantifier-free formulae) that are equivalent to these quantified formulae. The resulting QFFs include bounds for the horizon of uncertainty h and these bounds concern the robustness function $\widehat{\alpha}$ and the opportuneness function $\widehat{\beta}$ for the quantified formulae (3) and (4), respectively. Here we will also consider the more general mixed AE-case. The related universally–existentially (AE) quantified formula is presented in Subsection 2.3 below together with some remarks, which mainly concern the new immunity functions included in the QFFs (quantifier-free formulae) derived in the present applications.

2.2. Universally–existentially quantified formulae for solution sets of interval systems of equations

The present use of (mixed) universally–existentially (AE) quantified formulae in Ben-Haim's IGDT (info-gap or information-gap decision theory), see, e.g., [8, 10], was inspired by the frequent use of AE-quantified formulae in interval systems of linear algebraic equations with respect to their solution sets. Such systems are extremely well known in interval analysis having the general form

$$Ax = b \quad \text{with} \quad A \in \mathbf{A} \quad \text{and} \quad b \in \mathbf{b}. \tag{5}$$

In the above system, (5), A (the coefficient matrix) is an $m \times n$ matrix belonging to an interval matrix $\mathbf{A}$, b (the right-hand-side vector) is an m-vector belonging to an interval vector $\mathbf{b}$ and x is the n-vector of unknowns. An extensive related literature is available; see, e.g., the recent book by Skalna on parametric interval algebraic systems [94, Chapter 4, Subsections 4.1 and 4.2, pp. 85–88].

The related problems with universally–existentially (AE) quantified formulae for their solution sets include the following two classical problems for interval systems of linear algebraic equations:

(i) The interval tolerance problem and the related tolerable solution set; see, e.g., the papers by Nuding and Wilhelm [95], Neumaier [96], Kelling and Oelschlägel [97], Shary [98], Beaumont and Philippe [99] and Popova [100]. The solution set of the interval tolerance problem is defined as

$$\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})\ (\exists b \in \mathbf{b}) \text{ such that } Ax = b\}. \tag{6}$$

In this problem, the $m \times n$ elements A_{ij} of the coefficient matrix A are universally quantified whereas the m elements b_i of the right-hand-side vector b are existentially quantified.

(ii) The interval control problem and the related controllable solution set; see, e.g., the papers by Shary [101, 102] and Popova [103]. The solution set of the interval control problem is defined as

$$\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})\ (\exists A \in \mathbf{A}) \text{ such that } Ax = b\}. \tag{7}$$

Here, contrary to the tolerance problem, the m elements b_i of the right-hand-side vector b are universally quantified and the $m \times n$ elements A_{ij} of the coefficient matrix A are existentially quantified.

On the other hand, AE-solution sets of generalized problems related to interval linear systems were also investigated in detail; see, e.g., the papers by Shary [104, 105], Popova [106], Popova and Hladík [107] and Hladík [108]. In such problems, some elements of the matrix A and the vector b are universally quantified whereas the remaining elements of both the matrix A and the vector b are existentially quantified. Moreover, non-linear AE-solution sets were studied by Goldsztejn [109].

Of course, parametric interval linear systems (see, e.g., again the interesting book by Skalna [94] devoted to parametric interval algebraic systems), where the coefficient matrix A and the right-hand-side vector b in Eq. (5) depend on a vector $p = (p_1, p_2, \ldots, p_K)$ of K parameters with $p \in \mathbf{p}$ (with $\mathbf{p}$ being an interval K-vector) are also of particular importance especially with respect to the present results in the subsequent four sections, where parameters p_i are present in the corresponding models. Such systems were also studied by several authors; see, e.g., the papers (i) by Popova [100] for the parametric interval tolerance problem, (ii) by Popova [103] for the related control problem and (iii) by Popova and Krämer [110], Popova [106], Popova and Hladík [107] and Dehghani-Madiseh and Dehghan [111] for the general AE-problem, where some parameters p_j are universally quantified whereas the remaining parameters p_k are existentially quantified. This is exactly the case here with Ben-Haim's IGDT in its present modified form based on the use of AE-quantified formulae. Naturally, for parametric interval linear systems, where $A = A(p)$ and $b = b(p)$, Eq. (5) takes the modified form

$$A(p)x = b(p) \quad \text{with} \quad p \in \mathbf{p}. \tag{8}$$

It can also be mentioned that Popova [112] and Popova and Krämer [110] used (just for comparison purposes) the method of quantifier elimination in its implementation in *Mathematica* [68] together with the `Resolve` command for the determination of the solution sets of parametric interval linear systems without free parameters. The same approach was also recently used by the author, but it was based on the more appropriate (here) `Reduce` command instead of the `Resolve` command.

2.3. The (mixed) universally–existentially quantified formula (AE-quantified formula) in the IGDT

In this subsection, inspired by the use of (mixed) universally–existentially (AE) quantified formulae for interval systems of linear equations already briefly mentioned in the previous subsection, Subsection 2.2, we intend to generalize the quantified formulae (3) (universally quantified formula) and (4) (existentially quantified formula) of Subsection 2.1 concerning the IGDT to the mixed case, where both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) are present.

Here we assume that in the uncertainty problem under consideration (with a system model $\mathscr{M}$), we have a set of n uncertain variables q_i ($i = 1, 2, \ldots, n$ with $n \geq 2$) forming a vector q. Evidently, having adopted Ben-Haim's IGDT, these n uncertain variables q_i satisfy the assumed info-gap (or information-gap) model of uncertainty $\mathscr{U}$. The performance requirement(s) is (are) again denoted by the symbol $\mathscr{C}$. We also make again the related assumptions $\mathscr{A}$ mainly positivity assumptions.

In the present (mixed) universal–existential (AE) case of Ben-Haim's IGDT, we assume that the first n_1 uncertain variables q_j (with $1 \leq j \leq n_1$ and $1 \leq n_1 < n$) forming a vector q_{v1} are universally quantified. This means that in the quantified formula (9) below, the performance requirement(s) $\mathscr{C}$ should hold true for any set of acceptable values of these uncertain variables q_j (with $1 \leq j \leq n_1$), i.e. for any values of these uncertain variables simultaneously satisfying the system model $\mathscr{M}$ of the problem under consideration and the adopted info-gap model $\mathscr{U}$ in the IGDT of course always under the additional validity of the assumptions $\mathscr{A}$. Next, we assume that the remaining $n_2 = n - n_1$ (clearly, with $1 \leq n_2 < n$) uncertain variables q_k (with $n_1 + 1 \leq k \leq n$) forming a vector q_{v2} are existentially quantified. Hence, in the same quantified formula (9) below, the performance requirement(s) $\mathscr{C}$ should hold true for at least one set of acceptable values of these uncertain variables q_k analogously to the universally quantified variables, but now referring to *at least one* set of values.

Under the above circumstances here we have the generalized (mixed) AE-quantified formula

$\forall q_{v1} \, \exists q_{v2}$ in the info-gap model $\mathscr{U}$ such that if the system model $\mathscr{M}$ and the positivity assumptions $\mathscr{A}$ hold true, the performance requirement(s) $\mathscr{C}$ also holds (hold) true (9)

corresponding to the present mixed case. Here it is clear that beyond the $n_1 + n_2 = n$ quantified variables q_i in the above AE-quantified formula (our uncertain variables satisfying the assumed info-gap model $\mathscr{U}$) we also have the parameters of the problem. These are the free variables and will not be eliminated during quantifier elimination. We also have the uncertainty parameter (or horizon of uncertainty) h appearing in the info-gap model $\mathscr{U}$. This is also a free variable not to be eliminated.

Now by performing quantifier elimination to the above universally–existentially (AE) quantified formula (9), which includes both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists), we obtain a formula free from the n quantified uncertain variables q_i (with $1 \leq i \leq n$) that includes only the parameters p_l of the problem, which are free, non-quantified variables, and, additionally, the horizon of uncertainty h as well, which is also a free variable. This formula is the QFF (quantifier-free formula) that is valid for our uncertainty problem in Ben-Haim's IGDT. Usually, it is convenient to have the QFF solved with respect to the horizon of uncertainty h and this is easily achieved during quantifier elimination or just after it. Therefore, we are able to have available the bounds for the horizon of uncertainty h and this is our primary task here in the IGDT.

On the other hand, in the case of the universally quantified formula (3), we obtain a least upper bound (supremum, sup, usually also maximum) for the horizon of uncertainty h and this bound is the well-known robustness function (or simply robustness) $\widehat{\alpha}$ in the info-gap model under consideration. In this case, the horizon of uncertainty h should not exceed the robustness function $\widehat{\alpha}$ so that we can be sure that the performance requirement(s) $\mathscr{C}$ is (are) always satisfied under the validity of the assumed info-gap model $\mathscr{U}$ in the IGDT.

Analogously, in the case of the existentially quantified formula (4), we obtain a greatest lower bound (infimum, inf, usually also minimum) for the same horizon of uncertainty h and this bound is the well-known opportuneness function (or simply opportuneness) $\widehat{\beta}$ in the info-gap model under consideration. In this case, the horizon of uncertainty h should not be less than the opportuneness function $\widehat{\beta}$ so that the performance requirement(s) $\mathscr{C}$ is (are) satisfied by at least one set of values of the uncertain variables q_i under the validity of the assumed info-gap model $\mathscr{U}$ in the IGDT.

Related examples (in all cases obtained by employing the computational method of quantifier elimination [64] and *Mathematica* [68]) are presented in the applications of Ref. [91] as well as in the present applications in the four subsequent sections. Here it can also be mentioned that, naturally, frequently, the explicit expressions of the robustness and opportuneness functions $\widehat{\alpha}$ and $\widehat{\beta}$, respectively, are different for different ranges of the parameters p_l involved in the system model $\mathscr{M}$.

Here we are mainly interested in the mixed case with an AE-quantified formula of the form (9). In this general case, obviously, we obtain a QFF with a different immunity function, of course, not with the two classical robustness and opportuneness functions $\widehat{\alpha}$ and $\widehat{\beta}$, respectively. Additionally, in the same mixed case and in some problems under uncertainty conditions based on the info-gap model, the resulting QFFs may be more complicated than the ordinary QFFs corresponding to the universally and existentially quantified formulae (3) and (4), respectively, in the sense that we may have more than one immunity function related to the horizon of uncertainty h. These remarks are made clear in the applications of the four subsequent sections, where the mixed case is also studied.

More explicitly, in the application of Section 3 concerning the area of a rectangle, we obtain a new immunity function, $\widehat{\gamma}$, which is displayed in Eq. (32). We observe that this function, $\widehat{\gamma}$, corresponds to an infimum of the horizon of uncertainty h, i.e. $\widehat{\gamma} = \inf h$. This situation is analogous to what happens with the opportuneness function $\widehat{\beta}$, displayed (in this application) in Eq. (25). The opportuneness function $\widehat{\beta}$ also concerns an infimum of the horizon of uncertainty h, but in a completely different quantified formula (an existentially quantified formula) and, evidently, with a different analytical expression. Of course, this is a rather simple case in the mixed AE-quantified formulae studied here. Analogous is the second mixed case in Subsection 5.5 in the application of Section 5 concerning the volume of a rectangular cuboid, where the resulting QFF (quantifier-free formula) includes an immunity function $\widehat{\delta}$, Eq. (103), again concerning the infimum of the horizon of uncertainty h, i.e. $\widehat{\delta} = \inf h$. Completely similar is the mixed case in the application of Section 6 concerning the reactions at the ends of a fixed (clamped) ordinary beam, which is loaded by a concentrated load. In this case, the immunity function $\widehat{\varepsilon}$, Eq. (117), resulting in the derived QFF again concerns the infimum of the horizon of uncertainty h, i.e. $\widehat{\varepsilon} = \inf h$.

On the other hand, in the application of Section 4 concerning the buckling load of a fixed–free column, we will study two mixed cases that are related to two different AE-quantified formulae. In fact, in the second of these formulae, formula (63), the rôles of the universally and existentially quantified variables, L and EI, are reversed with respect to the first of these formulae, formula (57). In the first of these mixed cases, the case that corresponds to the AE-quantified formula (57), the resulting QFF includes an immunity function $\widehat{\gamma}$, displayed in Eq. (62), which concerns the infimum of the horizon of uncertainty h, i.e. $\widehat{\gamma} = \inf h$. But on the contrary, in the second of these mixed cases, the case that corresponds to the AE-quantified formula (63), the resulting QFF includes a new immunity function $\widehat{\delta}$, that displayed in Eq. (68), which concerns the supremum of the horizon of uncertainty h, i.e. $\widehat{\delta} = \sup h$. It is clear that this situation is completely different from the situation observed in the first application, that in Section 3, where, obviously, the reversion of the universally and existentially quantified variables has no influence at all on the type of the resulting immunity function. In fact, the same function $\widehat{\gamma}$ displayed in Eq. (32) is evidently obtained in both cases.

Finally, a really very interesting mixed case is the first mixed case (in Subsection 5.4) again in the application of Section 5 concerning the volume of a rectangular cuboid. In this particular mixed case, the resulting QFF, finally in the form (96), includes two immunity functions, $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$. Both of these immunity functions are defined in Eqs. (95). The first of these functions, the function $\widehat{\gamma}_1$, is a supremum of the uncertainty parameter (or horizon of uncertainty) h whereas the second of these functions, the function $\widehat{\gamma}_2$, is an infimum of the same parameter, the horizon of uncertainty h, of course under the validity of the whole QFF (96) and not independently for the functions $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$.

3. The area of a rectangle under uncertainty conditions

3.1. The system model, the info-gap model, the performance requirement and the assumptions

As a first and very simple application of the present approach based on Ben-Haim's info-gap model of uncertainty and quantifier elimination we consider the very simple problem of the area A of a rectangle of length a and width b. Both dimensions a and b of the rectangle are assumed to be uncertain variables and we adopt the ordinary fractional-error info-gap model for these variables. This very simple problem with two uncertain variables (the dimensions a and b of the rectangle) was already studied by the author [92, Section 4, pp. 15–16], but in a different case, that of two uncertainty parameters (or horizons of uncertainty) h_a and h_b. Yet, this study was made there in the usual purely universal case (AA-case or $\forall\forall$-case) in the quantified formula, i.e. in the case related to the robustness function with respect to the info-gap model. Evidently, completely analogous is the purely existential case (EE-case or $\exists\exists$-case), i.e. the case related to the opportuneness (or opportunity) function with respect to the same model. Here we consider the classical case of only one horizon of uncertainty h, but also taking into account the mixed universal–existential cases (AE-cases or $\forall\exists$-cases). This simple application will permit us to understand the use of the universal–existential cases in Ben-Haim's info-gap model with more than one uncertain variable (here a and b).

In the present simple problem, we use the following three components (elements) of the adopted model of uncertainty: (i) The system model $\mathscr{M}_r$ has the obvious form for the area A of a rectangle

$$\mathscr{M}_r := (A = ab). \tag{10}$$

(ii) Next, the info-gap model of uncertainty (uncertainty model) $\mathscr{U}_r$ is assumed to be a fractional-error info-gap model of course with respect to the uncertain variables (here dimensions) a and b. This info-gap model can be written in the final and more appropriate for quantifier elimination form

$$\mathscr{U}_r := (1-h)a_n \le a \le (1+h)a_n \wedge (1-h)b_n \le b \le (1+h)b_n. \tag{11}$$

Here h denotes the uncertainty parameter (or horizon of uncertainty) in this info-gap model and a_n and b_n denote the nominal values of the two dimensions a and b of the rectangle, respectively. (iii) Finally, the assumed performance requirement $\mathscr{C}_r$ is simply that the area $A = ab$ of the rectangle does not exceed a critical value A_c of this area A. Hence, we have the performance requirement

$$\mathscr{C}_r := A \le A_\mathrm{c} \quad \text{and, equivalently,} \quad \mathscr{C}_r := ab \le A_\mathrm{c}. \tag{12}$$

Additionally, we also make the following six positivity assumptions $\mathscr{A}_r$:

$$\mathscr{A}_r := a > 0 \wedge b > 0 \wedge a_n > 0 \wedge b_n > 0 \wedge h > 0 \wedge A_\mathrm{c} > 0. \tag{13}$$

All these positivity assumptions are obvious with the exception of the fifth assumption, $h > 0$, where for computational convenience we assumed the horizon of uncertainty h in the info-gap model $\mathscr{U}_r$ in Eq. (11) to be a positive variable and not simply a non-negative variable. Yet, this additional assumption (exclusion of the zero value for h) is not very important from the practical point of view.

Now we can formulate the present problem of uncertainty with the use of quantified formulae. Here we will use four such formulae related to (i) the (purely) universal (AA) case, (ii) the (purely) existential (EE) case and (iii) the two mixed universal–existential (AE) cases. These two cases are those on which we put emphasis here and they constitute the possible contribution of this technical report to Ben-Haim's info-gap uncertainty model and the relevant info-gap decision theory (IGDT).

3.2. The (purely) universal case related to the robustness function of the info-gap model

In this fundamental, extremely well-known and popular case, the quantified formula has the form

$\forall(a,b)$ in the info-gap model $\mathscr{U}_r$ such that the system model $\mathscr{M}_r$ and the positivity assumptions $\mathscr{A}_r$ hold true the performance requirement $\mathscr{C}_r$ also holds true. (14)

We observe that in this quantified formula only the universal quantifier $\forall$ (for all) is present. This means that this formula, (14), assures that the performance requirement $\mathscr{C}_r$ in Eq. (12) holds true for any pair (a,b) of values of the dimensions a and b of the rectangle under consideration provided, of course, that they satisfy the fractional-error info-gap model of uncertainty $\mathscr{U}_r$ defined in Eq. (11).

Evidently, now we can directly proceed to quantifier elimination here by continuously using its powerful implementation in *Mathematica*. The related quantifier elimination command has the form

```
Refine[Reduce[ForAll[{a,b}, Ur ∧ Ar, Cr], Reals], Ar]//Simplify        [c1]
```

with the symbol `Cr` here referring to the second form of the performance requirement $\mathscr{C}_r$ in Eq. (12). The resulting QFF (quantifier-free formula) has the very simple and, undoubtedly, expected form

$$A_c \geq (1+h)^2 a_n b_n \quad \text{or, equivalently,} \quad (1+h)^2 a_n b_n \leq A_c. \tag{15}$$

The reason that this form has been characterized as expected is simply that according to the info-gap model of uncertainty $\mathscr{U}_r$ defined in Eq. (11) the greatest value of the dimension a of the rectangle is $a_{max} = (1+h)a_n$ and, analogously, the greatest value of the dimension b of the same rectangle is $b_{max} = (1+h)b_n$. Therefore, the greatest value of the area $A = ab$ of the same rectangle is simply

$$A_{max} = a_{max} b_{max} = [(1+h)a_n][(1+h)b_n] = (1+h)^2 a_n b_n. \tag{16}$$

This is the worst case with respect to the performance requirement $\mathscr{C}_r$ in Eq. (12), which should be valid for all values of the two dimensions a and b of the rectangle. This happens because in this subsection we assume the validity of the universally quantified formula (14) related to the universal quantifier $\forall$ and, next, to the robustness function of our info-gap model $\mathscr{U}_r$. Hence, here we must have $A_{max} \leq A_c$ and thus we obtain the second form of the QFF (15) because of Eq. (16) for A_{max}.

Of course, here we are primarily interested in the values of the horizon of uncertainty h for which the performance requirement $\mathscr{C}_r$ in Eq. (12) holds true with our present info-gap model of uncertainty $\mathscr{U}_r$ in Eq. (11). This means that here we have to appropriately rewrite the QFF (15) by solving it with respect to the horizon of uncertainty h. This can be achieved by using the command

```
Refine[Reduce[qffAA, h], Ar]        [c2]
```

where the symbol `qffAA` denotes the QFF (15). Then we directly find that

$$h \leq \sqrt{\frac{A_c}{a_n b_n}} - 1 \quad \text{and, equivalently,} \quad h \leq \sqrt{\frac{A_c}{A_n}} - 1 \tag{17}$$

by using the substitution $A_n = a_n b_n$ for the nominal value A_n of the area $A = ab$ of the rectangle.

Naturally, in practice, the above QFF (17) can also be directly obtained by using instead of the original command [c1] the following slightly modified quantifier elimination command:

```
Refine[Reduce[ForAll[{a,b}, Ur ∧ Ar, Cr], h, Reals], Ar]        [c3]
```

where it is directly requested (by our having used the additional argument `h`) that the resulting QFF be solved with respect to the horizon of uncertainty h. This modified QFF has the following form:

$$a_n < \frac{A_c}{b_n} \wedge h \leq \sqrt{\frac{A_c}{a_n b_n}} - 1 \quad \text{and, equivalently,} \quad A_n < A_c \wedge h \leq \sqrt{\frac{A_c}{A_n}} - 1. \tag{18}$$

This QFF is essentially equivalent to the previous result (17) for the horizon of uncertainty h since its first conjunctive term, $a_n < A_c/b_n$ and, equivalently, $A_n < A_c$, simply denotes that the horizon of uncertainty h in its second conjunctive term can take positive values ($h > 0$). This was explicitly assumed in the fifth of our assumptions $\mathscr{A}_r$ in Eq. (13). On the other hand, it should be reminded that the above QFFs hold true only under the assumptions $\mathscr{A}_r$ in Eq. (13), which do not explicitly appear in the final forms of the QFFs because of the use of the `Refine` command with its second argument these assumptions $\mathscr{A}_r$. This is made here for the sake of a simpler appearance of any resulting QFF.

Finally, from our results (17) or, essentially equivalently, (18) it is clear that the robustness $\widehat{\alpha}$ of our info-gap model, i.e. the least upper bound (supremum, sup, here also maximum) of the horizon of uncertainty h, is

$$\widehat{\alpha} = \widehat{\alpha}(a_n, b_n, A_c) = \sup h = \sqrt{\frac{A_c}{a_n b_n}} - 1 = \sqrt{\frac{A_c}{A_n}} - 1 \quad \text{with} \ \ A_n < A_c. \tag{19}$$

This function $\widehat{\alpha}$, the popular robustness function in info-gap models of uncertainty, is the first and, surely, the most important function in the present uncertainty problem. (It is noted that, quite frequently, the symbols α and $\hat{h}$ are also used in lieu of the symbols h and $\widehat{\alpha}$ for the horizon of uncertainty and the robustness function, respectively.) Now we will proceed to the computation of the related opportuneness (or opportunity) function $\widehat{\beta}$. This is the second important function in the same uncertainty problem being related to the existential quantifier $\exists$ instead of the universal quantifier $\forall$.

3.3. The (purely) existential case related to the opportuneness function of the info-gap model

In this also interesting and very well-known case, the quantified formula to be used has the form

$\exists\,(a,b)$ in the info-gap model $\mathscr{U}_r$ such that if the system model $\mathscr{M}_r$ and the positivity assumptions $\mathscr{A}_r$ hold true, the performance requirement $\mathscr{C}_r$ also holds true. (20)

In this second quantified formula, only the existential quantifier $\exists$ (exists) is present contrary to the first quantified formula (14) in the previous subsection, where only the universal quantifier $\forall$ (for all) was present. This simply means that the above quantified formula, (20), assures that the performance requirement $\mathscr{C}_r$ in Eq. (12) holds true for at least one pair of values (a,b) of the dimensions a and b of the rectangle, of course, again provided that they satisfy the info-gap model $\mathscr{U}_r$ in Eq. (11).

We can again directly proceed to quantifier elimination continuing to employ its implementation in *Mathematica* [68]. The related quantifier elimination command now takes the existential form

```
Refine[Reduce[Exists[{a,b}, Ur ∧ Ar, Cr], Reals], Ar]//Simplify        [c4]
```

The resulting QFF (quantifier-free formula) has a very simple and again expected form. This form is

$$h \geq 1 \vee a_n \leq \frac{A_c}{(1-h)^2 b_n} \quad \text{or, equivalently,} \quad h \geq 1 \vee (1-h)^2 a_n b_n \leq A_c. \tag{21}$$

The reason that this form was also characterized as expected is simply that according to the info-gap model $\mathscr{U}_r$ in Eq. (11) the smallest value of the dimension a of the rectangle is $a_{\min} = (1-h)a_n$ and, analogously, the smallest value of the dimension b of the same rectangle is $b_{\min} = (1-h)b_n$. Therefore, the smallest value $A_{\min}$ of the area $A = ab$ of the present rectangle is determined as

$$A_{\min} = a_{\min} b_{\min} = [(1-h)a_n][(1-h)b_n] = (1-h)^2 a_n b_n. \tag{22}$$

This is the most favourable case with respect to the performance requirement $\mathscr{C}_r$ in Eq. (12), which should be valid for at least one pair of values of the dimensions a and b since in this subsection we assume the validity of the existentially quantified formula (20) related to the existential quantifier $\exists$ and, next, to the opportuneness (or opportunity) function for our info-gap model. Hence, we must have $A_{\min} \leq A_c$ and, finally, we obtain the second form of the QFF (21) because of Eq. (22) for $A_{\min}$.

Of course, here we are again primarily interested in the values of the uncertainty parameter (horizon of uncertainty) h for which the performance requirement $\mathscr{C}_r$ in Eq. (12) holds true for our present fractional-error info-gap model of uncertainty $\mathscr{U}_r$ defined in Eq. (11). This means that here we have to appropriately rewrite the QFF (21). This task can be achieved by using the command

```
Refine[Reduce[qffEE, h], Ar]//Simplify        [c5]
```

where the symbol `qffEE` denotes the aforementioned QFF (21). Then we directly find that

$$h \geq 1 \vee h \geq 1 - \sqrt{\frac{A_c}{a_n b_n}} \quad \text{and, equivalently,} \quad h \geq 1 \vee h \geq 1 - \sqrt{\frac{A_c}{A_n}} \tag{23}$$

after having used the substitution $A_n = a_n b_n$ for the nominal value A_n of the area A of the rectangle.

Evidently, in practice, the QFF (23) can, alternatively, be directly obtained simply by using the following slightly modified quantifier elimination command instead of the initial command [c4]:

```
Refine[Reduce[Exists[{a,b}, Ur ∧ Ar, Cr], h, Reals], Ar]                    [c6]
```

We observe that in this command it is directly requested that the resulting QFF appear solved with respect to the unknown uncertainty parameter (horizon of uncertainty) h. The new QFF has the form

$$a_n \leq \frac{A_c}{b_n} \vee \left(a_n > \frac{A_c}{b_n} \wedge h \geq 1 - \sqrt{\frac{A_c}{a_n b_n}}\right)$$

$$\text{and, equivalently,} \quad A_n \leq A_c \vee \left(A_n > A_c \wedge h \geq 1 - \sqrt{\frac{A_c}{A_n}}\right). \tag{24}$$

Finally, from both QFFs (23) and (24) it is directly concluded that in the present (purely) existential case in the quantified formula the opportuneness (or opportunity) function $\widehat{\beta}$, i.e. the greatest lower bound (infimum, inf, here also minimum) of the horizon of uncertainty h, is given by

$$\widehat{\beta} = \widehat{\beta}(a_n, b_n, A_c) = \inf h = 1 - \sqrt{\frac{A_c}{a_n b_n}} = 1 - \sqrt{\frac{A_c}{A_n}} \quad \text{with } A_n > A_c. \tag{25}$$

This function, the well-known opportuneness (or opportunity) function in info-gap models of uncertainty, is the second, but also sufficiently important, function in the present uncertainty problem.

Now we are ready to proceed with the computation of one more function relevant to our info-gap model of uncertainty $\mathscr{U}_r$ in the present application. This new function, $\widehat{\gamma}$, is related to both the universal quantifier $\forall$ and the existential quantifier $\exists$ and it now concerns the mixed (AE or $\forall\exists$) case.

3.4. The (mixed) universal–existential case and the related function of the info-gap model

In this mixed case, we consider the appearance of both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) contrary to the universally quantified formula (14) and to the existentially quantified formula (20) of the previous subsections. The quantified formula now has the form

$\forall a \exists b$ in the info-gap model $\mathscr{U}_r$ such that if the system model $\mathscr{M}_r$ and the positivity assumptions $\mathscr{A}_r$ hold true, the performance requirement $\mathscr{C}_r$ also holds true (26)

or the completely analogous form

$\forall b \exists a$ in the info-gap model $\mathscr{U}_r$ such that if the system model $\mathscr{M}_r$ and the positivity assumptions $\mathscr{A}_r$ hold true, the performance requirement $\mathscr{C}_r$ also holds true (27)

now with the order of the dimensions a and b of the rectangle reversed, i.e. with the dimension b universally quantified ($\forall b$) and the dimension a existentially quantified ($\exists a$). Here we will restrict our attention exclusively to the first quantified formula (26), the quantified formula with $\forall a$ and $\exists b$.

The above quantified formula (26) assures that the performance requirement $\mathscr{C}_r$ in Eq. (12) holds true for at least one value of the dimension b of the rectangle and this happens for any value of the other dimension a of the rectangle, of course, again provided that these dimensions a and b satisfy the assumed fractional-error info-gap model of uncertainty $\mathscr{U}_r$, which has been defined in Eq. (11).

Now we are ready to proceed with quantifier elimination again by using the general-purpose `Reduce` command of *Mathematica*. The related quantifier elimination command now has the form

```
Refine[Reduce[ForAll[a, Ur[[1]] ∧ Ar, Exists[b, Ur[[2]], Cr]], Reals], Ar]
        //Simplify                                                          [c7]
```

In this command, the symbol `Ur[[1]]` refers to the first conjunctive term in the info-gap model $\mathscr{U}_r$ in Eq. (11), i.e. to the term $(1-h)a_n \leq a \leq (1+h)a_n$ concerning the dimension a of the rectangle.

On the contrary, the symbol `Ur[[2]]` refers to the second conjunctive term in the same info-gap model $\mathscr{U}_r$ in Eq. (11), i.e. to the term $(1-h)b_n \le b \le (1+h)b_n$ concerning the dimension b of the same rectangle. The resulting QFF (quantifier-free formula) has the simple and again expected form

$$A_c + a_n b_n h^2 \ge a_n b_n \vee h \ge 1 \quad \text{and, equivalently,} \quad h \ge 1 \vee (1-h^2)a_n b_n \le A_c. \tag{28}$$

Now we will attempt to explain the reason that this form has been characterized as an expected form. In fact, according to the universally–existentially quantified formula (26) the performance requirement $\mathscr{C}_r$ in Eq. (12), i.e. $ab \le A_c$, should hold true for any value of the dimension a. (Presently, we do not pay attention to the values of the dimension b.) Therefore, this should happen for the greatest value of a, which is $a_{\max} = (1+h)a_n$ according to the info-gap model (11). Next, the same requirement $\mathscr{C}_r$ should also hold true for at least one value of the second dimension b and, clearly, the most favourable value of b for the validity of $\mathscr{C}_r$ is its smallest value, which is $b_{\min} = (1-h)b_n$ (provided that $h \le 1$) according to the same info-gap model (11). Hence, finally, we must have

$$a_{\max} b_{\min} = [(1+h)a_n][(1-h)b_n] \le A_c \quad \text{and, equivalently,} \quad (1-h^2)a_n b_n \le A_c. \tag{29}$$

This result constitutes an elementary verification of the relevant disjunctive term of the QFF (28). Naturally, if $h \ge 1$, then the greatest lower bound of b is zero (obviously, with $b > 0$). Therefore, the performance requirement $\mathscr{C}_r$ in Eq. (12) again holds true. Hence, the derived QFF (28) is correct.

Evidently, here we are again primarily interested in the values of the horizon of uncertainty h for which the performance requirement $\mathscr{C}_r$ in Eq. (12) holds true for our present info-gap model of uncertainty $\mathscr{U}_r$ in Eq. (11). This means that we have to appropriately rewrite the above QFF (28) by solving it with respect to h. This task can be easily achieved by using the supplementary command

```
Refine[Reduce[qffAE ∧ Ar, h]//PowerExpand, Ar]                    [c8]
```

with the symbol `qffAE` denoting the above QFF (28) and the positivity assumptions $\mathscr{A}_r$ in Eq. (13) continuously assumed to hold true. Then we obtain the modified QFF

$$\begin{aligned} & a_n \le \frac{A_c}{b_n} \vee \left(a_n > \frac{A_c}{b_n} \wedge h \ge \sqrt{1 - \frac{A_c}{a_n b_n}}\right) \\ \text{and, equivalently,} \quad & A_n \le A_c \vee \left(A_n > A_c \wedge h \ge \sqrt{1 - \frac{A_c}{A_n}}\right) \end{aligned} \tag{30}$$

again having used the substitution $A_n = a_n b_n$ for the nominal value A_n of the area A of the rectangle.

Naturally, in practice, the above QFF (30) can also be directly derived simply by using the following slightly modified quantifier elimination command instead of the initial command [c7]:

```
Refine[Reduce[ForAll[a, Ur[[1]] ∧ Ar, Exists[b, Ur[[2]], Cr]], h, Reals], Ar]
    //Simplify                                                     [c9]
```

where it has been directly requested that the resulting QFF appear solved with respect to the horizon of uncertainty h through the addition of the argument `h`. The resulting QFF now has the form

$$A_c \ge a_n b_n \vee h \ge \sqrt{1 - \frac{A_c}{a_n b_n}} \quad \text{and, equivalently,} \quad A_n \le A_c \vee h \ge \sqrt{1 - \frac{A_c}{A_n}}, \tag{31}$$

which is essentially equivalent to the QFF (30) with the assumptions $\mathscr{A}_r$ again assumed to hold true.

We observe that the greatest lower bound $\widehat{\gamma}$ (the infimum, inf, here also the minimum) of the horizon of uncertainty h is

$$\widehat{\gamma} = \widehat{\gamma}(a_n, b_n, A_c) = \inf h = \sqrt{1 - \frac{A_c}{a_n b_n}} = \sqrt{1 - \frac{A_c}{A_n}}. \tag{32}$$

This new function, $\widehat{\gamma}$, is the function of interest in this subsection and it is a third function related to the present info-gap model beyond the two classical functions: (i) the robustness function $\widehat{\alpha}$ derived in Subsection 3.2 and (ii) the opportuneness (or opportunity) function $\widehat{\beta}$ derived in Subsection 3.3.

3.5. The (mixed) universal–existential case with two horizons of uncertainty in the info-gap model

In quite a similar manner, we can study the non-classical case concerning two horizons of uncertainty (the horizon of uncertainty h_a concerning the dimension a of the rectangle and the horizon of uncertainty h_b concerning its dimension b) in the assumed info-gap model of uncertainty $\mathscr{U}_r$ defined in Eq. (11) with one horizon of uncertainty h there, i.e. with the same horizon of uncertainty for both dimensions a and b of the rectangle. Then this info-gap model $\mathscr{U}_r$ takes the modified form

$$\mathscr{U}_{rm} := (1-h_a)a_n \le a \le (1+h_a)a_n \wedge (1-h_b)b_n \le b \le (1+h_b)b_n. \tag{33}$$

In this case, the positivity assumptions $\mathscr{A}_r$ in Eq. (13) take their modified form (now with h_a and h_b)

$$\mathscr{A}_{rm} := a > 0 \wedge b > 0 \wedge a_n > 0 \wedge b_n > 0 \wedge h_a > 0 \wedge h_b > 0 \wedge A_\mathrm{c} > 0. \tag{34}$$

In this subsection, we restrict our attention to the (mixed, AE or $\forall\exists$) universal–existential case in the quantified formula that will be used here. The (purely) universal case has also been recently studied in Ref. [92, Section 4, pp. 15–16]. Here we assume the validity of the related universally–existentially quantified formula (26). This quantified formula now takes the following slightly modified form (now simply with $\mathscr{U}_{rm}$ instead of $\mathscr{U}_r$ and, additionally, with $\mathscr{A}_{rm}$ instead of $\mathscr{A}_r$):

$\forall a\, \exists b$ in the info-gap model $\mathscr{U}_{rm}$ such that if the system model $\mathscr{M}_r$ and the positivity assumptions $\mathscr{A}_{rm}$ hold true, the performance requirement $\mathscr{C}_r$ also holds true. (35)

(The case of the quantified formula (27) with the order of the two dimensions a and b of the rectangle reversed, i.e. with the dimension a existentially quantified and the dimension b universally quantified, is completely similar and, therefore, there is no reason at all to consider it here.)

Now in order to perform quantifier elimination to the above quantified formula (35) we can use the quantifier elimination command [c7], but now, obviously, with `Urm` instead of `Ur` for the info-gap model of uncertainty and `Arm` instead of `Ar` for the assumptions. Then we obtain the simple QFF

$$A_\mathrm{c} + (h_a+1)(h_b-1)a_n b_n \ge 0 \vee h_b \ge 1 \quad \text{and, equivalently,} \quad (1+h_a)(1-h_b)A_n \le A_\mathrm{c} \vee h_b \ge 1 \tag{36}$$

because we have $A_n = a_n b_n$ for the nominal value A_n of the area A of the present rectangle as has been already mentioned.

The interpretation of the above QFF (36) is again very simple and analogous to that made in the previous subsection below Eq. (28) in the case of a single horizon of uncertainty h. Here the only difference is that the maximum value a_max of the dimension a of the rectangle and the minimum value b_min of the dimension b of the rectangle are given by $a_\mathrm{max} = (1+h_a)a_n$ and $b_\mathrm{min} = (1-h_b)b_n$, respectively. Therefore, Eq. (29) in the previous subsection now takes the slightly generalized form

$$a_\mathrm{max}\, b_\mathrm{min} = [(1+h_a)a_n][(1-h_b)b_n] \le A_\mathrm{c} \quad \text{and, equivalently,} \quad (1+h_a)(1-h_b)A_n \le A_\mathrm{c} \tag{37}$$

in agreement with the second form of the above QFF (36).

Of course, it is very easy to solve the above QFF (36) with respect to h_a or h_b naturally taking into consideration that the modified positivity assumptions $\mathscr{A}_{rm}$ defined in Eq. (34) should continuously hold true. Alternatively, we can use the following modified quantifier elimination command, which is again based on the `Reduce` general-purpose command of *Mathematica*:

```
Refine[Reduce[ForAll[a, Urm[[1]] ∧ Arm, Exists[b, Urm[[2]], Cr]], {ha,hb},
         Reals], Arm]//Simplify//Apart                                    [c10]
```

now with the additional argument `{ha,hb}` in the `Reduce` command. Then we get the QFF

$$h_b \ge 1 - \frac{A_\mathrm{c}}{(1+h_a)a_n b_n} \vee \left(A_\mathrm{c} > a_n b_n \wedge h_a \le \frac{A_\mathrm{c}}{a_n b_n} - 1\right) \tag{38}$$

and, equivalently, since we have $A_n = a_n b_n$ for the nominal value A_n of the area A of the rectangle,

$$h_b \geq 1 - \frac{A_c}{(1+h_a)A_n} \vee \left(A_c > A_n \wedge h_a \leq \frac{A_c}{A_n} - 1\right). \tag{39}$$

Another equivalent, but somewhat more complicated in appearance, QFF has also been obtained simply by selecting the order `{hb,ha}` instead of the order `{ha,hb}` in the above command [c10].

It is understood that the above-derived QFFs hold true under the positivity assumptions $\mathscr{A}_{rm}$ in Eq. (34) although these assumptions are not explicitly included in these QFFs because of the use of the `Refine` command with argument these assumptions $\mathscr{A}_{rm}$ for the derivation of simpler QFFs.

4. The buckling load of a fixed–free column under uncertainty conditions

4.1. The system model, the info-gap model, the performance requirement and the assumptions

As a second application of the present methodology based on (i) the info-gap model (but here with the additional study of the two mixed universal–existential cases beyond the classical purely universal and purely existential cases) and (ii) quantifier elimination in this section we will study the problem of buckling of a simple column of length L fixed (clamped) at its lower end $x = 0$ and free at its upper end $x = L$. This is the famous buckling problem originally studied and solved by Euler.

For this problem under uncertainty here our first task is simply to determine or define the three components of our model, i.e. (i) the system model $\mathscr{M}_b$, (ii) the info-gap model $\mathscr{U}_b$ and (iii) the performance requirement $\mathscr{C}_b$. Additionally, we will also make six positivity assumptions $\mathscr{A}_b$. Next, we will consider the four related uncertainty cases concerning (i) the (purely) universal, (ii) the (purely) existential and (iii) the two (mixed) universal–existential quantified formulae proceeding to quantifier eliminations and thus determining the related functions of interest. Here the emphasis is put on the (mixed) universal–existential formulae. Here it should be mentioned that the purely universal formula was recently studied in Ref. [92] in the case of two horizons of uncertainty, h_{EI} and h_L, i.e. in the case of a separate horizon of uncertainty for each uncertain variable. Here we will restrict our attention to one horizon of uncertainty, h, but we will put an emphasis on the (mixed) universal–existential cases not having been studied so far. Naturally, the whole approach is similar to that followed in the previous section, Section 3, for the area $A = ab$ of a rectangle of dimensions a and b.

At first, the buckling load (or Euler's critical load) P_b of the present fixed–free column under buckling conditions is determined by the classical Euler formula (see, e.g., [113, p. 48, Eq. (2-4)])

$$P_b = \frac{\pi^2 EI}{4L^2}. \tag{40}$$

In this formula, the symbol EI denotes the (least, minimum) flexural rigidity of the column. As is well known, the flexural rigidity EI is equal to the product of the modulus of elasticity (or Young's modulus) E of the isotropic elastic material of the column and the (least, minimum) moment of inertia I of the cross-section of the column. Here the flexural rigidity EI of the column is assumed to be a single physical–geometrical quantity as is quite frequently the case. Therefore, here the system model $\mathscr{M}_b$ for the present fixed–free column under buckling conditions has the form

$$\mathscr{M}_b := \left(P_b = \frac{\pi^2 EI}{4L^2}\right). \tag{41}$$

Here it is assumed that both the (least, minimum) flexural rigidity EI and the length L of the present fixed–free column are uncertain variables (or uncertain parameters) that have the nominal values EI_n and L_n, respectively. Moreover, these uncertain variables are assumed to satisfy an info-gap uncertainty model $\mathscr{U}_b$ of the usual fractional-error type. This info-gap model, $\mathscr{U}_b$, is written below in its final conjunctive logical form (without fractions) appropriate for quantifier elimination

$$\mathscr{U}_b := (1-h)EI_n \leq EI \leq (1+h)EI_n \wedge (1-h)L_n \leq L \leq (1+h)L_n. \tag{42}$$

Here the new symbol h denotes the uncertainty parameter (or horizon of uncertainty) concerning the uncertain (least, minimum) flexural rigidity EI and the uncertain length L of the present column under buckling conditions in the above info-gap model $\mathscr{U}_b$ defined in Eq. (42). This horizon of uncertainty, h, is here assumed to be applicable to both of these uncertain variables, EI and L.

Next, of course, the performance requirement $\mathscr{C}_b$ assumed in the present buckling problem is simply that the buckling load P_b determined by Eq. (40) should not exceed an appropriate critical buckling load P_c. Hence, the performance requirement $\mathscr{C}_b$ should have the form of the inequality constraint

$$\mathscr{C}_b := P_b \leq P_c \quad \text{and, equivalently,} \quad \mathscr{C}_b := \frac{\pi^2 EI}{4L^2} \leq P_c, \tag{43}$$

where the system model $\mathscr{M}_b$ in Eq. (41) was already taken into account in the second form of $\mathscr{C}_b$.

Finally, we also make the obvious six positivity assumptions

$$\mathscr{A}_b := EI > 0 \wedge L > 0 \wedge EI_n > 0 \wedge L_n > 0 \wedge h > 0 \wedge P_c > 0. \tag{44}$$

For computational convenience here we again assume the positivity (instead of the non-negativity) of the horizon of uncertainty h concerning both the uncertain (least, minimum) flexural rigidity EI and the uncertain length L of the column in the assumed info-gap model $\mathscr{U}_b$ defined in Eq. (42).

4.2. The (purely) universal case related to the robustness function of the info-gap model

In this fundamental, popular and really very important case, the quantified formula has the form

$\forall (EI, L)$ in the info-gap model $\mathscr{U}_b$ such that the system model $\mathscr{M}_b$ and the positivity assumptions $\mathscr{A}_b$ hold true the performance requirement $\mathscr{C}_b$ also holds true. (45)

In the above quantified formula, only the universal quantifier $\forall$ (for all) is present. This means that this formula, (45), assures that the performance requirement $\mathscr{C}_b$ defined in Eq. (43) holds true for any pair of values (EI, L) of the uncertain flexural rigidity EI and the similarly uncertain length L of the present fixed–free column provided, of course, that both these uncertain variables, EI and L, satisfy the info-gap model of uncertainty $\mathscr{U}_b$ defined in Eq. (42).

Of course, now we can again directly proceed to quantifier elimination by using its powerful implementation in *Mathematica* [68] continuously preferring the use of the general-purpose `Reduce` command instead of the `Resolve` command because the latter command may lead (fortunately in rare cases) to complicated QFFs. The related quantifier elimination command has the simple form

```
Refine[Reduce[ForAll[{EI,L}, Ub ∧ Ab, Cb], Reals], Ab]//Simplify        [c11]
```

The resulting QFF (quantifier-free formula) has the very simple and again expected form

$$h < 1 \wedge P_c \geq \frac{\pi^2 (1+h) EI_n}{4(1-h)^2 L_n^2}. \tag{46}$$

The reason that the above QFF was characterized as expected is simply that in accordance with the info-gap model $\mathscr{U}_b$ defined in Eq. (42) the performance requirement $\mathscr{C}_b$ defined in Eq. (43) should hold true for all pairs of values (EI, L) of the flexural rigidity EI and the length L of the present column that satisfy the info-gap model $\mathscr{U}_b$. Hence, we can consider only the related worst case. Evidently, because of the system model $\mathscr{M}_b$ determined by Eq. (41) or, almost equivalently, the second form of the performance requirement $\mathscr{C}_b$ defined in Eq. (43) the worst case is that where the flexural rigidity EI takes its maximum value $EI_{\max} = (1+h)EI_n$ and the length L takes its minimum value $L_{\min} = (1-h)L_n$, where, as was already mentioned, EI_n and L_n denote the nominal values of the uncertain variables EI and L, respectively. Therefore, in order that the performance requirement $\mathscr{C}_b$ holds true for all acceptable values of the uncertain variables EI and L we must have

$$\frac{\pi^2 EI_{\max}}{4L_{\min}^2} \leq P_c \quad \text{and, finally,} \quad \frac{\pi^2 (1+h) EI_n}{4(1-h)^2 L_n^2} \leq P_c \tag{47}$$

in agreement with the above QFF (46). The additional inequality constraint $h < 1$ in this QFF, (46), essentially concerns the requirement of positive values of the length L of the column (as well as of its flexural rigidity EI) according to the adopted info-gap model $\mathscr{U}_b$ having been defined in Eq. (42).

Naturally, here we are particularly interested in the values of the horizon of uncertainty h for which the quantified formula (45) holds true. These values of h become clear by using the command

```
Refine[Reduce[qffAA, h, Ab]//Apart                                [c12]
```

Here the new symbol `qffAA` simply denotes the above QFF (46). The resulting final form of this QFF (but now solved with respect to the uncertainty parameter or horizon of uncertainty h) is

$$h \leq 1 + \frac{\pi^2 EI_n - \pi\sqrt{EI_n(\pi^2 EI_n + 32L_n^2 P_\mathrm{c})}}{8L_n^2 P_\mathrm{c}}. \tag{48}$$

Of course, alternatively, it is completely possible to directly derive an equivalent form of the above final result (48) simply by using the slightly modified quantifier elimination command

```
Refine[Reduce[ForAll[{EI,L}, Ub ∧ Ab, Cb], h, Reals], Ab]//Together      [c13]
```

(now with the use of the additional argument `h` in the `Reduce` command and, next, the use of the auxiliary command `Together` instead of the auxiliary command `Simplify`) or the similar command with the use of the auxiliary command `Apart` instead of the auxiliary command `Together`. In both cases, the resulting QFFs (quantifier-free formulae) can be directly written in their final form

$$EI_n < \frac{4L_n^2 P_\mathrm{c}}{\pi^2} \wedge h \leq 1 + \frac{\pi^2 EI_n - \pi\sqrt{EI_n(\pi^2 EI_n + 32L_n^2 P_\mathrm{c})}}{8L_n^2 P_\mathrm{c}}. \tag{49}$$

Now we observe that the first conjunctive term in the above QFF (49) concerns only the nominal values EI_n and L_n of the flexural rigidity EI and the length L of the column, respectively. This term simply assures the existence of a positive horizon of uncertainty h in the QFF (46) as has been already assumed in the assumptions $\mathscr{A}_b$ defined in Eq. (44). It is repeated and completely understood that all the above QFFs hold true only under the validity of these positivity assumptions $\mathscr{A}_b$ simply because of the use of the `Refine` command with second argument these assumptions $\mathscr{A}_b$. The use of the `Refine` command permits a simpler and generally preferable appearance of the resulting QFF.

Finally, we mention that as is clear from the previous and final QFF (49), the least upper bound (supremum, sup, here also maximum) $\widehat{\alpha}$ of the horizon of uncertainty h is given by the formula

$$\widehat{\alpha} = \widehat{\alpha}(EI_n, L_n, P_\mathrm{c}) := \sup h = 1 + \frac{\pi^2 EI_n - \pi\sqrt{EI_n(\pi^2 EI_n + 32L_n^2 P_\mathrm{c})}}{8L_n^2 P_\mathrm{c}}. \tag{50}$$

This is the robustness function (or simply the robustness) $\widehat{\alpha} = \widehat{\alpha}(EI_n, L_n, P_\mathrm{c})$ in the assumed info-gap model of uncertainty that concerns an ordinary fixed–free column under buckling conditions. This function, $\widehat{\alpha}$, is extremely well-known and, surely, it is of particular importance (in fact, it is the most important function) in Ben-Haim's IGDT (info-gap or information-gap decision theory) [6–8].

4.3. The (purely) existential case related to the opportuneness function of the info-gap model

In quite a similar manner, we can also work with the (purely) existential case, which is related to the opportuneness (or opportunity) function of the info-gap model $\mathscr{U}_b$ defined in Eq. (42). In this also interesting and well-known case in Ben-Haim's IGDT, the quantified formula takes the form

$\exists\,(EI, L)$ in the info-gap model $\mathscr{U}_b$ such that if the system model $\mathscr{M}_b$ and the positivity assumptions $\mathscr{A}_b$ hold true, the performance requirement $\mathscr{C}_b$ also holds true. (51)

In the above quantified formula, only the existential quantifier $\exists$ (exists) is present. This means that this formula, (51), assures that the performance requirement $\mathscr{C}_b$ in Eq. (43) holds true for at least one pair of values (EI, L) of the flexural rigidity EI and the length L of the present column under buckling conditions provided, of course, that they satisfy the info-gap model $\mathscr{U}_b$ defined in Eq. (42).

Evidently, now we can again directly proceed to quantifier elimination by employing its powerful implementation in *Mathematica* [68]. The related quantifier elimination command has the form

```
Refine[Reduce[Exists[{EI,L}, Ub ∧ Ab, Cb], Reals], Ab]//Simplify        [c14]
```

The resulting QFF (quantifier-free formula) has the very simple and again expected final form

$$h \geq 1 \vee \frac{\pi^2(1-h)EI_n}{4(1+h)^2L_n^2} \leq P_c. \tag{52}$$

Once more the reason that the above form (52) has been characterized as expected is simply that in accordance with the info-gap model $\mathscr{U}_b$ defined in Eq. (42) the performance requirement $\mathscr{C}_b$ defined in Eq. (43) should hold true for at least one pair of values (EI, L) of the flexural rigidity EI and the length L of the present fixed–free column satisfying the info-gap model $\mathscr{U}_b$. Hence, here we can consider only the related most favourable case. Naturally, because of the system model $\mathscr{M}_b$ determined by Eq. (41) or, equivalently, the second form of the performance requirement $\mathscr{C}_b$ defined in Eq. (43) the most favourable case is the case where the flexural rigidity EI of the column takes its minimum value $EI_{\min} = (1-h)EI_n$ and its length L takes its maximum value $L_{\max} = (1+h)L_n$, where, as has been already mentioned, EI_n and L_n denote the two related nominal values. Therefore, in order that the performance requirement $\mathscr{C}_b$ defined in Eq. (43) holds true for at least one pair of values (EI, L) of the two uncertain parameters EI and L of the present column we must have

$$\frac{\pi^2 EI_{\min}}{4L_{\max}^2} \leq P_c \quad \text{and, finally,} \quad \frac{\pi^2(1-h)EI_n}{4(1+h)^2L_n^2} \leq P_c. \tag{53}$$

This result is in complete agreement with the second disjunctive term in the above QFF (52). On the other hand, the inequality constraint $h \geq 1$ in the same QFF essentially concerns the possibility of zero or even negative values of the flexural rigidity EI of the column. This possibility, $EI \leq 0$, is physically unacceptable and, therefore, this inequality constraint, $h \geq 1$, is practically not important.

Of course, we are particularly interested in the values of the horizon of uncertainty h for which the existentially quantified formula (51) holds true. These values are found by using the command

```
Refine[Reduce[qffEE, h, Ab]//Apart        [c15]
```

where the new symbol `qffEE` denotes the above QFF (52). The resulting final form of this QFF, which is now solved with respect to the unknown horizon of uncertainty h, has the form

$$h \geq 1 \vee h \geq -1 + \frac{-\pi^2 EI_n + \pi\sqrt{EI_n(\pi^2 EI_n + 32L_n^2P_c)}}{8L_n^2P_c}. \tag{54}$$

Clearly, it is also completely possible to directly derive an essentially equivalent form of the above final result (54) simply by using the slightly modified quantifier elimination command

```
Refine[Reduce[Exists[{EI,L}, Ub ∧ Ab, Cb], h, Reals], Ab]//Together        [c16]
```

now with the use of the additional argument `h` in the `Reduce` command, which is here again used for quantifier elimination. The resulting QFF can be directly written in its final form

$$EI_n \leq \frac{4L_n^2P_c}{\pi^2} \vee \left[EI_n > \frac{4L_n^2P_c}{\pi^2} \wedge h \geq -1 + \frac{-\pi^2 EI_n + \pi\sqrt{EI_n(\pi^2 EI_n + 32L_n^2P_c)}}{8L_n^2P_c}\right]. \tag{55}$$

Here we observe that the first disjunctive term in the above QFF (55) concerns the two nominal values EI_n and L_n of the flexural rigidity EI and the length L of the column, respectively, and it assures the validity of the performance requirement $\mathscr{C}_b$ defined in Eq. (43) for these two nominal values. Hence, in this case, the existentially quantified formula (51) holds true simply because this happens at least for the aforementioned nominal values EI_n and L_n. The second and, surely, more significant possibility is the opposite case, i.e. the case of validity of the first conjunctive term of the second disjunctive term in the above QFF (55) instead of the first disjunctive term in this QFF. In this case, the second conjunctive term of the second disjunctive term in this QFF should hold true for the validity of the existentially quantified formula (51). Hence, we must have $h \geq \widehat{\beta}$, where

$$\widehat{\beta} = \widehat{\beta}(EI_n, L_n, P_c) := \inf h = -1 + \frac{-\pi^2 EI_n + \pi\sqrt{EI_n(\pi^2 EI_n + 32L_n^2 P_c)}}{8L_n^2 P_c}. \tag{56}$$

This quantity, $\widehat{\beta} = \widehat{\beta}(EI_n, L_n, P_c)$, is the greatest lower bound (infimum, inf, here also minimum) of the horizon of uncertainty h in the present (purely) existential case. This bound is the very well-known opportuneness function or opportunity function (or simply opportuneness) in the present info-gap model of uncertainty $\mathscr{U}_b$ defined in Eq. (42) and concerning a fixed−free column under buckling conditions. This is the second important immunity function in Ben-Haim's IGDT (info-gap or information-gap decision theory) [6–8] after the robustness function $\widehat{\alpha} = \widehat{\alpha}(EI_n, L_n, P_c)$ already having been computed here for the present uncertainty problem and determined by Eq. (50).

Now we can proceed to two additional cases beyond the (purely) universal case studied in the previous subsection and the (purely) existential case studied in this subsection. These additional cases are the two mixed (AE or $\forall\exists$) cases with the simultaneous appearance of both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) in the related quantified formulae concerning the two uncertain parameters EI and L of the present column. These two cases, which constitute the possible contribution of the present technical report to the famous Ben-Haim's IGDT (info-gap or information-gap decision theory) [6–8], will be studied in the next two subsections.

4.4. The first mixed universal−existential case and the related function of the info-gap model

In this subsection, we consider the first mixed case here with the appearance of the universal quantifier $\forall$ with respect to the flexural rigidity EI of the column and the existential quantifier $\exists$ with respect to the length L of the column contrary to the universally quantified formula (45) and the existentially quantified formula (51) studied in the previous two subsections. The second mixed case (the case with the appearance of the universal quantifier with respect to the length L and the existential quantifier with respect to the flexural rigidity EI) will be studied in the next subsection.

In the present first mixed case, the universally−existentially quantified formula has the form

$\forall EI\, \exists L$ in the info-gap model $\mathscr{U}_b$ such that if the system model $\mathscr{M}_b$ and the positivity assumptions $\mathscr{A}_b$ hold true, the performance requirement $\mathscr{C}_b$ also holds true (57)

here with the flexural rigidity EI universally quantified and the length L existentially quantified.

The above quantified formula, (57), assures us that the performance requirement $\mathscr{C}_b$ in Eq. (43) holds true for at least one value (existential quantifier $\exists$) of the length L of the column and that this happens for any value (universal quantifier $\forall$) of its flexural rigidity EI again here provided, of course, that both of these uncertain variables EI and L satisfy the info-gap model $\mathscr{U}_b$ in Eq. (42).

Now we again proceed to quantifier elimination in order to compute the related QFF (quantifier-free formula). The related quantifier elimination command that is used here has the following form:

```
Refine[Reduce[ForAll[EI,Ub[[1]] ∧ Ab, Exists[L, Ub ∧ Ab, Cb]], Reals],
     Ab]//Simplify                                                   [c17]
```

This command is again based on the `Reduce` general-purpose command of *Mathematica*, which is continuously used here instead of the `Resolve` command devoted to quantifier elimination. The resulting QFF has the form

$$P_c \geq \frac{\pi^2 EI_n}{4(1+h)L_n^2} \quad \text{or, equivalently,} \quad \frac{\pi^2 EI_n}{4(1+h)L_n^2} \leq P_c. \tag{58}$$

Now by using the supplementary command

```
Refine[Reduce[qffAE, h], Ab]//Apart
```
[c18]

with the new symbol `qffAE` denoting the above QFF (58), we obtain its final form that now appears solved with respect to the unknown horizon of uncertainty h of interest here. This is the form

$$h \geq \frac{\pi^2 EI_n}{4L_n^2 P_c} - 1. \tag{59}$$

Naturally, an essentially equivalent form of the latter form (59) of the QFF (58) can alternatively be directly obtained by using the slightly modified quantifier elimination command

```
Refine[Reduce[ForAll[EI, Ub[[1]] ∧ Ab, Exists[L, Ub ∧ Ab, Cb]], h, Reals],
        Ab]//Apart
```
[c19]

In this command, the additional (optional) argument `h` in the `Reduce` command assures the appearance of the resulting QFF in a form solved with respect to the unknown horizon of uncertainty h. This QFF has the form

$$EI_n \leq \frac{4L_n^2 P_c}{\pi^2} \vee \left(EI_n > \frac{4L_n^2 P_c}{\pi^2} \wedge h \geq \frac{\pi^2 EI_n}{4L_n^2 P_c} - 1 \right). \tag{60}$$

It is understood that here all QFFs presuppose the validity of the positivity assumptions $\mathscr{A}_b$ defined in Eq. (44), that include the positivity assumption $h > 0$ made here for the horizon of uncertainty h.

The interpretation of the fundamental (although not solved with respect to h) QFF (58) is not very difficult. In fact, we wish that the performance requirement $\mathscr{C}_b$ in the universally–existentially quantified formula (57) should hold true here for any value (because of the presence of the universal quantifier $\forall$) of the flexural rigidity EI of the column satisfying the info-gap model of uncertainty $\mathscr{U}_b$ defined in Eq. (42). Next, because of the system model $\mathscr{M}_b$ determined by Eq. (41), this should happen for the greatest value $EI_{\max} = (1+h)EI_n$ of the flexural rigidity EI. (This greatest value is clear from the info-gap model $\mathscr{U}_b$ defined in Eq. (42).) On the other hand, from the same quantified formula, (57), it is also clear that this should happen for at least one value of the length L satisfying the info-gap model $\mathscr{U}_b$ defined in Eq. (42). Naturally, for this purpose it is sufficient that we select the most favourable value of L and this is its maximum value $L_{\max} = (1+h)L$ again because of the info-gap model $\mathscr{U}_b$. (This is clear because of the appearance of L in the denominator of the fraction in the system model $\mathscr{M}_b$ determined by Eq. (41).) Under these circumstances here we must have

$$\frac{\pi^2 EI_{\max}}{4L_{\max}^2} = \frac{\pi^2 (1+h) EI_n}{4(1+h)^2 L_n^2} = \frac{\pi^2 EI_n}{4(1+h)L_n^2} \leq P_c. \tag{61}$$

Hence, we have derived the QFF (58) in its second form. Undoubtedly, the solution of this inequality constraint with respect to the unknown horizon of uncertainty h is very easy and, in this way, we directly obtain the inequality constraint (59) or, equivalently, the last term in the QFF (60).

From the inequality constraint (59) as well as from the related QFF (60) we directly observe that the greatest lower bound (infimum, inf, here also minimum) of the horizon of uncertainty h is

$$\widehat{\gamma} = \widehat{\gamma}(EI_n, L_n, P_c) := \inf h = \frac{\pi^2 EI_n}{4L_n^2 P_c} - 1. \tag{62}$$

This is the new function concerning the present universally–existentially quantified formula (57) as a third function beyond the classical robustness function $\widehat{\alpha} = \widehat{\alpha}(EI_n, L_n, P_c)$ in Eq. (50) concerning the purely universal case and, next, the well-known opportuneness function $\widehat{\beta} = \widehat{\beta}(EI_n, L_n, P_c)$ in Eq. (56) concerning the purely existential case of course both related to the present info-gap model of uncertainty. Finally, a fourth related function, $\widehat{\delta} = \widehat{\delta}(EI_n, L_n, P_c)$, concerning the second mixed universal–existential case in the quantified formula with the rôles of the two parameters EI (flexural rigidity) and L (length) of the present column reversed will be computed in the next subsection.

4.5. The second mixed universal–existential case and the related function of the info-gap model

Quite similarly, we can study the second mixed universal–existential case in the present buckling problem for a fixed–free column and find the related function of the info-gap model. The present case again concerns the appearance of both the universal and the existential quantifiers in the quantified formula, but now with the rôles of the parameters EI and L reversed as was already mentioned. Then the quantified formula (57) in the previous subsection takes the slightly modified form

$$\forall L\,\exists EI \quad \text{in the info-gap model } \mathscr{U}_b \text{ such that if the system model } \mathscr{M}_b \text{ and the positivity assumptions } \mathscr{A}_b \text{ hold true, the performance requirement } \mathscr{C}_b \text{ also holds true} \tag{63}$$

now with the length L of the column universally quantified and its flexural rigidity EI existentially quantified contrary to the quantified formula (57) in the previous subsection, Subsection 4.4.

Here the whole approach is quite similar to that in the previous subsection for the first mixed universal–existential case having been studied there. It is understood that now the above quantified formula, (63), assures that the performance requirement $\mathscr{C}_b$ in Eq. (43) holds true for at least one value (existential quantifier $\exists$) of the flexural rigidity EI of the column and that this happens for any value (universal quantifier $\forall$) of its length L again here provided, of course, that both of these parameters, EI and L, satisfy the assumed info-gap model of uncertainty $\mathscr{U}_b$ defined in Eq. (42).

Working analogously to the previous subsection by using the quantifier elimination command

```
Refine[Reduce[ForAll[L,Ub[[2]] ∧ Ab, Exists[EI, Ub ∧ Ab, Cb]], Reals],
        Ab]//Simplify                                                    [c20]
```

we directly obtain with the help of *Mathematica* [68] the related QFF (quantifier-free formula)

$$h \geq 1 \vee P_{\mathrm{c}} \geq \frac{\pi^2 EI_n}{4(1-h)L_n^2} \quad \text{or, equivalently,} \quad h \geq 1 \vee \frac{\pi^2 EI_n}{4(1-h)L_n^2} \leq P_{\mathrm{c}}. \tag{64}$$

Now by solving the above QFF with respect to the unknown horizon of uncertainty h (either by hand or, alternatively, by using again the `Reduce` command of *Mathematica*) we find its equivalent form

$$h \geq 1 \vee h \leq 1 - \frac{\pi^2 EI_n}{4L_n^2 P_{\mathrm{c}}}. \tag{65}$$

Alternatively, by using the slightly modified quantifier elimination command

```
Refine[Reduce[ForAll[L,Ub[[2]] ∧ Ab, Exists[EI, Ub ∧ Ab, Cb]], h, Reals],
        Ab]//Apart                                                       [c21]
```

now with the additional argument `h` in the `Reduce` command and without the auxiliary `Simplify` command, but with the auxiliary `Apart` command instead, we obtain the essentially equivalent QFF

$$\left[EI_n < \frac{4L_n^2 P_{\mathrm{c}}}{\pi^2} \wedge \left(h \geq 1 \vee h \leq 1 - \frac{\pi^2 EI_n}{4L_n^2 P_{\mathrm{c}}}\right)\right] \vee \left(h \geq 1 \wedge EI_n \geq \frac{4L_n^2 P_{\mathrm{c}}}{\pi^2}\right). \tag{66}$$

Evidently, the case $h \geq 1$, which permits the appearance of zero and negative values of the two uncertain variables EI and L in the info-gap model U_b having been defined in Eq. (42), has no significant practical importance. Here the really important term in the above QFF (66) is the term

$$h \leq 1 - \frac{\pi^2 EI_n}{4L_n^2 P_{\mathrm{c}}} \quad \text{and, equivalently,} \quad h \leq \widehat{\delta} = \widehat{\delta}(EI_n, L_n, P_{\mathrm{c}}). \tag{67}$$

The new symbol $\widehat{\delta} = \widehat{\delta}(EI_n, L_n, P_{\mathrm{c}})$ denotes a new function concerning the present mixed universal–existential case. Clearly, this function is defined as

$$\widehat{\delta} = \widehat{\delta}(EI_n, L_n, P_{\mathrm{c}}) := \sup h = 1 - \frac{\pi^2 EI_n}{4L_n^2 P_{\mathrm{c}}} \tag{68}$$

and it concerns the least upper bound (supremum, sup, here also maximum) of the unknown horizon of uncertainty h. This new function is analogous to the three functions $\widehat{\alpha} = \widehat{\alpha}(EI_n, L_n, P_{\mathrm{c}})$ in Eq. (50), $\widehat{\beta} = \widehat{\beta}(EI_n, L_n, P_{\mathrm{c}})$ in Eq. (56) and $\widehat{\gamma} = \widehat{\gamma}(EI_n, L_n, P_{\mathrm{c}})$ in Eq. (62) of the previous three subsections.

5. The volume of a rectangular cuboid under uncertainty conditions

5.1. The system model, the info-gap model, the performance requirement and the assumptions

As a third application of the present methodology based on (i) Ben-Haim's info-gap model of uncertainty and (ii) quantifier elimination in this section we consider the case of the volume V of a rectangular cuboid (rectangular or orthogonal parallelepiped) of dimensions a (length), b (width) and c (height). Here all these three dimensions are assumed to be uncertain variables (or uncertain parameters). Now, at first, we have to define the three components (elements) of the info-gap model, i.e. (i) the system model $\mathscr{M}_v$, (ii) the info-gap model of uncertainty $\mathscr{U}_v$ and (iii) the performance requirement $\mathscr{C}_v$. The related necessary positivity assumptions $\mathscr{A}_v$ are also defined. The present results concern a problem with three uncertain variables (or uncertain parameters) and, additionally, they generalize the results of Section 3 concerning the area A of a rectangle of dimensions a and b.

At first, as far as the system model $\mathscr{M}_v$ is concerned, obviously, this model has the extremely well-known and elementary form of the product of the three dimensions a, b and c of the present rectangular cuboid, i.e.

$$\mathscr{M}_v := (V = abc), \tag{69}$$

where, as has been already mentioned, the symbol V denotes the volume of the rectangular cuboid.

Next, the assumed info-gap model $\mathscr{U}_v$ is again the fractional-error model here with respect to the three uncertain dimensions a, b and c and with the same uncertainty parameter (or horizon of uncertainty) h. The final form of this info-gap model $\mathscr{U}_v$ (appropriate for quantifier elimination) is

$$\mathscr{U}_v := (1-h)a_n \le a \le (1+h)a_n \wedge (1-h)b_n \le b \le (1+h)b_n \wedge (1-h)c_n \le c \le (1+h)c_n. \tag{70}$$

In this info-gap model $\mathscr{U}_v$, the symbols a_n, b_n and c_n denote the nominal values of the length a, the width b and the height c of the rectangular cuboid, respectively. Obviously, the nominal value V_n of the volume V of the rectangular cuboid is the product of the nominal values a_n, b_n and c_n of its dimensions a, b and c, respectively, i.e.

$$V_n = a_n b_n c_n. \tag{71}$$

Next, the performance requirement $\mathscr{C}_v$ assumed in the present uncertainty model is simply that the volume $V = abc$ of the rectangular cuboid does not exceed a critical value V_c of this volume, i.e.

$$\mathscr{C}_v := V \le V_\mathrm{c} \quad \text{and, equivalently,} \quad \mathscr{C}_v := abc \le V_\mathrm{c}, \tag{72}$$

where the system model $\mathscr{M}_v$ in Eq. (69) has been taken into account in the second form of $\mathscr{C}_v$.

Finally, we also make eight positivity assumptions $\mathscr{A}_v$. These positivity assumptions are

$$\mathscr{A}_v := a > 0 \wedge b > 0 \wedge c > 0 \wedge a_n > 0 \wedge b_n > 0 \wedge c_n > 0 \wedge h > 0 \wedge V_\mathrm{c} > 0. \tag{73}$$

Simply for computational reasons, in the seventh of these assumptions, $h > 0$, we again preferred to assume the positivity of the uncertainty parameter (or horizon of uncertainty) h instead of its non-negativity, $h \ge 0$. Nevertheless, this assumption (the additional exclusion of the value zero for the horizon of uncertainty h, i.e. the additional assumption $h \ne 0$) is of minor practical importance.

5.2. The (purely) universal case related to the robustness function of the info-gap model

We begin with the classical (purely) universal case, where all three dimensions a, b and c of the rectangular cuboid, i.e. the present three uncertain variables, are universally quantified. In this fundamental and really very important case, the related quantified formula has the form

$\forall(a,b,c)$ in the info-gap model $\mathscr{U}_v$ such that the system model $\mathscr{M}_v$ and the positivity assumptions $\mathscr{A}_v$ hold true the performance requirement $\mathscr{C}_v$ also holds true. (74)

In this quantified formula, only the universal quantifier $\forall$ (for all) is present. This means that this formula, (74), assures that the performance requirement $\mathscr{C}_v$ defined in Eq. (72) holds true for any triplet of values (a,b,c) of the dimensions a, b and c of the rectangular cuboid naturally provided that these dimensions satisfy the info-gap model of uncertainty $\mathscr{U}_v$ defined in Eq. (70). The present (purely) universal case was also recently studied by the author in Ref. [92, Section 5, pp. 17–18] in the case of three horizons of uncertainty (h_a, h_b and h_c), but without dimension parameters, i.e. with numerical values ($a_n = 2$, $b_n = 8$ and $c_n = 5$) assigned to the three nominal values a_n, b_n and c_n.

Evidently, here we can again directly proceed to quantifier elimination by employing its powerful implementation in *Mathematica* [68]. The related quantifier elimination command has the form

```
Refine[Reduce[ForAll[{a,b,c}, Uv ∧ Av, Cv], Reals], Av]//Simplify        [c22]
```

The resulting QFF (quantifier-free formula) has the very simple and, undoubtedly, expected form

$$V_c \geq (1+h)^3 a_n b_n c_n \quad \text{and, equivalently,} \quad (1+h)^3 V_n \leq V_c \tag{75}$$

because of Eq. (71) for the nominal value $V_n = a_n b_n c_n$ of the volume $V = abc$ of the rectangular cuboid. This is a completely expected result because here we wish that the quantified formula (74) holds true for all values of the three dimensions a, b and c of the rectangular cuboid of course here with the restriction that they satisfy the info-gap model of uncertainty $\mathscr{U}_v$ defined in Eq. (70). Hence, they should satisfy the performance requirement $\mathscr{C}_v$ defined in Eq. (72). Therefore, this requirement should hold true even in the worst case and here it is clear that the worst case is the case of the greatest values $a_{\max} = (1+h)a_n$, $b_{\max} = (1+h)b_n$ and $c_{\max} = (1+h)c_n$ (according to the info-gap model $\mathscr{U}_v$ defined in Eq. (70)) of the three dimensions a, b and c, respectively. With this simple way of thinking we directly derive the above QFF (75) in the present (purely) universal case.

Clearly, the same QFF can be solved with respect to the horizon of uncertainty h. The result is

$$h \leq \sqrt[3]{\frac{V_c}{V_n}} - 1 \quad \text{and, equivalently,} \quad h \leq \left(\frac{V_c}{V_n}\right)^{1/3} - 1 \tag{76}$$

and we also have already assumed that $h > 0$. Obviously, $V_c/V_n \geq 1$ because here we wish that the performance requirement $\mathscr{C}_v$ defined in Eq. (72) holds true for all values of a, b and c satisfying the info-gap model $\mathscr{U}_v$ defined in Eq. (70). Consequently, this requirement should also hold true for their nominal values a_n, b_n and c_n, respectively. Hence, $V_n = a_n b_n c_n \leq V_c$ and, therefore, $V_c/V_n \geq 1$.

On the other hand, by using the modified quantifier elimination command

```
Refine[Reduce[ForAll[{a,b,c}, Uv ∧ Av, Cv], h, Reals], Av]        [c23]
```

instead of the previous command [c22], that is essentially by adding the argument `h` to the `Reduce` command and not using the `Simplify` command any more, we again find the QFF (75), but now solved with respect to the horizon of uncertainty h. The resulting QFF can be written in the form

$$a_n b_n c_n < V_c \land h \leq r_{1,1} \quad \text{and, equivalently,} \quad V_n < V_c \land h \leq r_{1,1} \tag{77}$$

because $V_n = a_n b_n c_n$. In this QFF, the symbol $r_{1,1}$ denotes the first root of the cubic polynomial

$$p_1(s) := a_n b_n c_n (s^3 + 3s^2 + 3s + 1) - V_c = a_n b_n c_n (s+1)^3 - V_c = V_n (s+1)^3 - V_c \tag{78}$$

again because $V_n = a_n b_n c_n$. The root $r_{1,1}$ can be directly computed either manually or, alternatively, by using the well-known `Solve` command of *Mathematica* and it is given by the following formula:

$$r_{1,1} = \sqrt[3]{\frac{V_c}{a_n b_n c_n}} - 1 = \sqrt[3]{\frac{V_c}{V_n}} - 1 \quad \text{and, equivalently,} \quad r_{1,1} = \left(\frac{V_c}{a_n b_n c_n}\right)^{1/3} - 1 = \left(\frac{V_c}{V_n}\right)^{1/3} - 1. \tag{79}$$

From the above QFF (77), which is accompanied by Eqs. (78) and (79), we directly observe that the well-known robustness function (or, simply, the robustness) $\widehat{\alpha}$ of the assumed info-gap model of uncertainty $\mathscr{U}_v$ defined in Eq. (70) has the form

$$\widehat{\alpha} = \widehat{\alpha}(a_n, b_n, c_n, V_c) = \sup h = r_{1,1} = \sqrt[3]{\frac{V_c}{a_n b_n c_n}} - 1 = \sqrt[3]{\frac{V_c}{V_n}} - 1. \tag{80}$$

As is well known, this is the least upper bound (the supremum, sup, here also the maximum) of the horizon of uncertainty h in the present application related to the volume V of a rectangular cuboid.

Now we are ready to proceed to the related (purely) existential case in the assumed quantified formula and, next, to the two (mixed) universal–existential (AE or $\forall\exists$) cases in the same formula.

5.3. The (purely) existential case related to the opportuneness function of the info-gap model

In this subsection, we proceed to the consideration of the (purely) existential case in the quantified formula. In this case, all three dimensions a, b and c of the rectangular cuboid (the present uncertain variables) are existentially quantified. Then the related quantified formula takes the form

$\exists(a,b,c)$ in the info-gap model $\mathscr{U}_v$ such that if the system model $\mathscr{M}_v$ and the positivity assumptions $\mathscr{A}_v$ hold true, then the performance requirement $\mathscr{C}_v$ also holds true. (81)

In the above quantified formula, only the existential quantifier $\exists$ (exists) is present. This simply means that this formula, (81), assures that the performance requirement $\mathscr{C}_v$ defined in Eq. (72) holds true for at least one triplet of values (a,b,c) of the three dimensions a, b and c of the rectangular cuboid naturally provided that they satisfy the info-gap model of uncertainty $\mathscr{U}_v$ defined in Eq. (70).

Evidently, now we can again directly proceed to quantifier elimination by using its powerful implementation in *Mathematica* [68]. The related quantifier elimination command has the form

```
Refine[Reduce[Exists[{a,b,c}, Uv ∧ Av, Cv], Reals], Av]//Simplify        [c24]
```

now with the existential quantifier `Exists` instead of the universal quantifier `ForAll` in the previous subsection. The resulting QFF (quantifier-free formula) has the simple and surely expected form

$$h \geq 1 \vee a_n \leq \frac{V_\mathrm{c}}{(1-h)^3 b_n c_n} \quad \text{and, equivalently,} \quad h \geq 1 \vee (1-h)^3 V_n \leq V_\mathrm{c}. \tag{82}$$

The second form of this QFF holds true because the nominal value V_n of the volume V of the rectangular cuboid is here determined by the simple formula $V_n = a_n b_n c_n$ according to Eq. (71).

Naturally, the above QFF (82) is a completely expected result. This happens because here we wish that the quantified formula (81) holds true for at least one triplet of values (a,b,c) of the uncertain dimensions a, b and c of the rectangular cuboid of course with the restriction that these dimensions satisfy the info-gap model of uncertainty $\mathscr{U}_v$ in Eq. (70). It is clear that the most favourable case that permits the validity of the performance requirement $\mathscr{C}_v$ in Eq. (72) is the case where the three uncertain dimensions a, b and c take their smallest values $a_\mathrm{min} = (1-h)a_n$, $b_\mathrm{min} = (1-h)b_n$ and $c_\mathrm{min} = (1-h)c_n$, respectively. In this case, the existentially quantified formula (81) holds true. Hence, we must have

$$a_\mathrm{min} b_\mathrm{min} c_\mathrm{min} = (1-h)^3 a_n b_n c_n = (1-h)^3 V_n \leq V_\mathrm{c}. \tag{83}$$

This result is in agreement with the above QFF (82) concerning the present (purely) existential case.

Of course, the second and most important disjunctive term in the QFF (82) can be directly solved with respect to the horizon of uncertainty h appearing in this term. The result is

$$h \geq 1 - \sqrt[3]{\frac{V_\mathrm{c}}{V_n}} = 1 - \left(\frac{V_\mathrm{c}}{V_n}\right)^{1/3}. \tag{84}$$

Alternatively, the same result can be derived by using the modified quantifier elimination command

```
Refine[Reduce[Exists[{a,b,c}, Uv ∧ Av, Cv], h, Reals], Av]        [c25]
```

with the additional argument `h` in the `Reduce` command, which assures that the resulting QFF will appear solved with respect to the unknown horizon of uncertainty h. This QFF can be written as

$$a_n b_n c_n \leq V_\mathrm{c} \vee (a_n b_n c_n > V_\mathrm{c} \wedge h \geq r_{2,1}) \quad \text{and, equivalently,} \quad V_n \leq V_\mathrm{c} \vee (V_n > V_\mathrm{c} \wedge h \geq r_{2,1}) \tag{85}$$

because $V_n = a_n b_n c_n$. In this QFF, the symbol $r_{2,1}$ denotes the first root of the cubic polynomial

$$p_2(s) := a_n b_n c_n (s^3 - 3s^2 + 3s - 1) + V_\mathrm{c} = a_n b_n c_n (s-1)^3 + V_\mathrm{c} = V_n (s-1)^3 + V_\mathrm{c} \tag{86}$$

again because $V_n = a_n b_n c_n$. This root, $r_{2,1}$, can be directly computed (either manually or, alternatively, by using the well-known `Solve` command of *Mathematica*) and it is given by the formula

$$r_{2,1} = 1 - \sqrt[3]{\frac{V_c}{a_n b_n c_n}} = 1 - \sqrt[3]{\frac{V_c}{V_n}} \quad \text{and, equivalently,} \quad r_{2,1} = 1 - \left(\frac{V_c}{a_n b_n c_n}\right)^{1/3} = 1 - \left(\frac{V_c}{V_n}\right)^{1/3}. \quad (87)$$

From the above QFF (85), which is accompanied by Eq. (86), that defines the polynomial $p_2(s)$, and by Eq. (87) for its first root $r_{2,1}$, we directly conclude that the opportuneness function (or, simply, the opportuneness) $\widehat{\beta}$ of the assumed info-gap model of uncertainty $\mathscr{U}_v$ in Eq. (70) has the form

$$\widehat{\beta} = \widehat{\beta}(a_n, b_n, c_n, V_c) = \inf h = r_{2,1} = 1 - \sqrt[3]{\frac{V_c}{a_n b_n c_n}} = 1 - \sqrt[3]{\frac{V_c}{V_n}}. \quad (88)$$

As is well known, this is the greatest lower bound (the infimum, inf, here also the minimum) of the horizon of uncertainty h in the present application related to the volume V of a rectangular cuboid.

Now we are ready to proceed to the two (mixed) universal–existential (AE or $\forall\exists$) cases in the quantified formula. These are the cases that are of particular interest in the present technical report.

5.4. The universal–existential case with a, b universally quantified and c existentially quantified

In this subsection, we proceed to the consideration of the (mixed, AE or $\forall\exists$) universal–existential case in the quantified formula used with two uncertain variables universally quantified (here the dimensions a and b of the rectangular cuboid) and one uncertain variable existentially quantified (here the dimension c of the same cuboid). In this case, the related quantified formula takes the form

$\forall(a,b)\,\exists c$ in the info-gap model $\mathscr{U}_v$ such that if the system model $\mathscr{M}_v$ and the positivity assumptions $\mathscr{A}_v$ hold true, then the performance requirement $\mathscr{C}_v$ also holds true. (89)

In the above quantified formula, both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) are present. This means that this formula, (89), assures that the performance requirement $\mathscr{C}_v$ in Eq. (72) holds true for at least one value of the dimension c of the rectangular cuboid and this happens for any pair of values (a,b) of the dimensions a and b of the same cuboid of course provided that these three dimensions a, b and c satisfy the info-gap model $\mathscr{U}_v$ defined in Eq. (70).

Naturally, now we can again directly proceed to quantifier elimination by using its powerful implementation in *Mathematica* [68]. The related quantifier elimination command now has the form

```
Refine[Reduce[ForAll[{a,b}, Uv[[1]] ∧ Uv[[2]] ∧ Av, Exists[c, Uv ∧ Av, Cv]],
        Reals], Av]//Simplify                                          [c26]
```

with the use of the universal quantifier `ForAll` with respect to the first two dimensions a and b of the rectangular cuboid and, next, the use of the existential quantifier `Exists` with respect to the third dimension c of the same cuboid instead of the universal quantifier `ForAll` with respect to all three dimensions a, b and c in Subsection 5.2 (concerning the purely universal case) and the existential quantifier $\exists$ with respect to the same dimensions in Subsection 5.3 (concerning the purely existential case). Here the resulting QFF (quantifier-free formula) has the very simple and again expected form

$$h \geq 1 \vee -(1-h)(1+h)^2 a_n b_n c_n + V_c \geq 0 \quad \text{and, equivalently,} \quad h \geq 1 \vee (1-h)(1+h)^2 V_n \leq V_c. \quad (90)$$

In the second and more convenient form of this QFF, it was again taken into account that the nominal value V_n of the volume V of the present rectangular cuboid is $V_n = a_n b_n c_n$ according to Eq. (71).

Of course, the above QFF (90) is again a completely expected result. This happens simply because here we wish that the quantified formula (89) holds true for all pairs of values (a,b) of the two dimensions a and b of the rectangular cuboid under consideration evidently with the restriction that these two dimensions satisfy the info-gap model of uncertainty $\mathscr{U}_v$ defined in Eq. (70). Therefore, we should consider the related worst case and this is the case when these two dimensions a and b

take their maximum values $a_{\max} = (1+h)a_n$ and $b_{\max} = (1+h)b_n$, respectively. Additionally, the same quantified formula, (89), should also hold true for at least one value of the dimension c of the same cuboid and clearly the most favourable related case permitting the validity of the performance requirement $\mathscr{C}_v$ in Eq. (72) is that when this dimension takes its minimum value $c_{\min} = (1-h)c_n$. Under these conditions with respect to the present info-gap model $\mathscr{U}_v$ in Eq. (70) we must have

$$a_{\max} b_{\max} c_{\min} = [(1+h)a_n][(1+h)b_n][(1-h)c_n] = (1-h)(1+h)^2 a_n b_n c_n$$
$$= (1-h)(1+h)^2 V_n \leq V_c \tag{91}$$

since $V_n = a_n b_n c_n$. Clearly, this result for the present mixed case is in agreement with the QFF (90).

On the other hand, because here we are mainly interested in the values of the horizon of uncertainty h, we can alternatively use the following slightly modified quantifier elimination command:

```
Refine[Reduce[ForAll[{a,b}, Uv[[1]] ∧ Uv[[2]] ∧ Av, Exists[c, Uv ∧ Av, Cv]],
         h, Reals], Av]//Simplify                                        [c27]
```

In this command, we have added the argument `h` to the basic `Reduce` command so that the resulting QFF be solved with respect to the horizon of uncertainty h. This QFF can be written in the final form

$$(a_n b_n c_n < V_c \wedge h \leq r_{3,2}) \vee h \geq r_{3,3} \vee 32 a_n b_n c_n \leq 27 V_c \tag{92}$$

and, equivalently (because the nominal value V_n of the volume V of the cuboid is $V_n = a_n b_n c_n$),

$$(V_n < V_c \wedge h \leq r_{3,2}) \vee h \geq r_{3,3} \vee 32 V_n \leq 27 V_c \tag{93}$$

and it consists of three disjunctive terms. In this QFF, (92) or, equivalently, (93), the two new symbols $r_{3,2}$ and $r_{3,3}$ denote the second and the third real roots, respectively, of the cubic polynomial

$$p_3(s) := (s^3 + s^2 - s - 1) a_n b_n c_n + V_c = (s-1)(s+1)^2 V_n + V_c. \tag{94}$$

The somewhat strange and, probably, interesting situation in the QFF (92) and, equivalently, (93) is that in the present (mixed) universal–existential case we have two functions $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ related to the present info-gap model of uncertainty defined in Subsection 5.1. Clearly, these are the functions

$$\widehat{\gamma}_1 = \widehat{\gamma}_1(a_n, b_n, c_n, V_c) := r_{3,2} \quad \text{and} \quad \widehat{\gamma}_2 = \widehat{\gamma}_2(a_n, b_n, c_n, V_c) := r_{3,3}, \tag{95}$$

where the two symbols $r_{3,2}$ and $r_{3,3}$ were already defined. Now, using this notation, we can rewrite the QFF (93) in the form

$$(V_n < V_c \wedge h \leq \widehat{\gamma}_1) \vee h \geq \widehat{\gamma}_2 \vee 32 V_n \leq 27 V_c. \tag{96}$$

Naturally, we observe that here the first new function $\widehat{\gamma}_1 = \widehat{\gamma}_1(a_n, b_n, c_n, V_c)$ serves as a least upper bound (supremum, sup, here also maximum) of the horizon of uncertainty h in the first disjunctive term of the above QFF (96). On the contrary, the second new function $\widehat{\gamma}_2 = \widehat{\gamma}_2(a_n, b_n, c_n, V_c)$ serves as a greatest lower bound (infimum, inf, here also minimum) of the same horizon of uncertainty h, but now in the second disjunctive term of the same QFF (96).

It is clear that the above situation is more complicated than the situations already observed in the two previous subsections, where we had only one immunity function related to the adopted model of uncertainty. More explicitly, in Subsection 5.2 we had the robustness function $\widehat{\alpha} = \widehat{\alpha}(a_n, b_n, c_n, V_c)$ (defined in Eq. (80)) whereas in Subsection 5.3 we had the opportuneness (or opportunity) function $\widehat{\beta} = \widehat{\beta}(a_n, b_n, c_n, V_c)$ (defined in Eq. (88)). As is extremely well-known, both these immunity functions, $\widehat{\alpha}$ and $\widehat{\beta}$, are classical functions in Ben-Haim's IGDT (info-gap decision theory). On the contrary, the present additional functions, $\widehat{\gamma}_1 = \widehat{\gamma}_1(a_n, b_n, c_n, V_c)$ and $\widehat{\gamma}_2 = \widehat{\gamma}_2(a_n, b_n, c_n, V_c)$, are nonclassical functions and, probably, new functions related to the same info-gap model. Evidently, this happens since $\widehat{\alpha}$ and $\widehat{\beta}$ refer to the classical (purely) universal and existential cases, respectively, in the quantified formula whereas the present functions $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ refer to a (mixed) AE (or $\forall\exists$) case.

Finally, we can add that in this subsection we restricted our attention to the universal–existential case $\forall(a,b)\,\exists c$ in the assumed quantified formula (89). Evidently, completely analogous and of no additional interest are the two cases $\forall(b,c)\,\exists a$ and $\forall(c,a)\,\exists b$ leading again to the same functions $\widehat{\gamma}_1 = \widehat{\gamma}_1(a_n,b_n,c_n,V_c)$ and $\widehat{\gamma}_2 = \widehat{\gamma}_2(a_n,b_n,c_n,V_c)$ defined in Eqs. (95) and again related to the uncertainty parameter (or horizon of uncertainty) h. Therefore, there is no need at all to consider these additional but also completely similar cases $\forall(b,c)\,\exists a$ and $\forall(c,a)\,\exists b$ and we will omit them.

On the contrary, it is of interest to study the case where only one uncertain dimension, e.g. the dimension a, of the rectangular cuboid is universally quantified in the quantified formula whereas the remaining two uncertain dimensions, e.g. here the dimensions b and c, are existentially quantified in the same quantified formula. This interesting and rather important additional (mixed, AE or $\forall\exists$) case in the present uncertainty problem will be studied in some detail in the next subsection.

5.5. The universal–existential case with a universally quantified and b, c existentially quantified

In this last subsection of the present section, we proceed to the consideration of the (mixed, AE or $\forall\exists$) universal–existential case in the quantified formula, but now with one uncertain variable (here the dimension a of the rectangular cuboid) universally quantified and two uncertain variables (here the dimensions b and c of the same cuboid) existentially quantified. In this second (mixed, AE or $\forall\exists$) case, the related universally–existentially quantified formula takes the following form:

$\forall a\,\exists(b,c)$ in the info-gap model $\mathscr{U}_v$ such that if the system model $\mathscr{M}_v$ and the positivity assumptions $\mathscr{A}_v$ hold true, then the performance requirement $\mathscr{C}_v$ also holds true. (97)

In this quantified formula, both the universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (exists) are again present. Here this means that this formula, (97), assures us that the performance requirement $\mathscr{C}_v$ in Eq. (72) holds true for at least one pair of values (b,c) of the dimensions b and c of the rectangular cuboid and this happens for any value of the dimension a of the same cuboid evidently provided that these three dimensions a, b and c satisfy the info-gap model $\mathscr{U}_v$ in Eq. (70).

Of course, now we can again directly proceed to quantifier elimination by continuing to use its implementation in *Mathematica* [68]. The related quantifier elimination command now has the form

```
Refine[Reduce[ForAll[a, Uv[[1]] ∧ Av, Exists[{b,c}, Uv ∧ Av, Cv]],
        Reals], Av]//Simplify                                    [c28]
```

Here the universal quantifier `ForAll` concerns only the first dimension a of the rectangular cuboid whereas the existential quantifier `Exists` concerns the second and the third dimensions b and c, respectively, of the same cuboid contrary to the first mixed case, in which the universal quantifier `ForAll` concerns the two dimensions a and b of the cuboid and the existential quantifier `Exists` concerns the third dimension c of the cuboid. This case was already studied in the previous Subsection 5.4. Here the resulting QFF (quantifier-free formula) has the simple and again expected form

$$h \geq 1 \vee (1-h)^2(1+h)\,a_n b_n c_n \leq V_c \quad \text{and, equivalently,} \quad h \geq 1 \vee (1-h)^2(1+h)V_n \leq V_c \qquad (98)$$

of course, again because the nominal value V_n of the volume V of the present rectangular cuboid is the product of the three nominal values a_n, b_n and c_n, i.e. $V_n = a_n b_n c_n$, in accordance with Eq. (71).

Naturally, the above QFF (98) is again a completely expected result because here we wish that the universally–existentially quantified formula (97) holds true for all values of the dimension a of the rectangular cuboid under consideration evidently with the restriction that this dimension a satisfies the info-gap model of uncertainty $\mathscr{U}_v$ in Eq. (70). Therefore, we should consider the related worst case and this is the case where this dimension a takes its maximum value $a_{\max} = (1+h)a_n$. Moreover, the same quantified formula, (97), should hold true for at least one pair of values (b,c) of the dimensions b and c of the same cuboid. The most favourable related case that permits the validity of the performance requirement $\mathscr{C}_v$ in Eq. (72) is that where these dimensions b and c take their minimum values $b_{\min} = (1-h)b_n$ and $c_{\min} = (1-h)c_n$, respectively. Under these conditions with respect to the present info-gap model of uncertainty $\mathscr{U}_v$ defined in Eq. (70) we must have

$$\begin{aligned} a_{\max} b_{\min} c_{\min} &= [(1+h)a_n][(1-h)b_n][(1-h)c_n] = (1-h)^2(1+h)\,a_n b_n c_n \\ &= (1-h)^2(1+h)V_n \le V_c \end{aligned} \tag{99}$$

because $V_n = a_n b_n c_n$. Of course, this result for the present mixed case agrees with the QFF (98).

Alternatively and completely analogously to the three previous subsections, we can also use the slightly modified quantifier elimination command

```
Refine[Reduce[ForAll[a, Uv[[1]] ∧ Av, Exists[{b,c}, Uv ∧ Av, Cv]],
        h, Reals], Av]//Simplify                                  [c29]
```

Here we again added the argument `h` to the basic `Reduce` command so that the resulting QFF can appear solved with respect to the horizon of uncertainty h. This QFF can be written in its final form

$$a_n b_n c_n \le V_c \vee (a_n b_n c_n > V_c \wedge h \ge r_{4,2}) \tag{100}$$

and, equivalently (because the nominal value V_n of the volume V of the cuboid is $V_n = a_n b_n c_n$),

$$V_n \le V_c \vee (V_n > V_c \wedge h \ge r_{4,2}). \tag{101}$$

Here we have only two disjunctive terms contrary to the previous subsection, Subsection 5.4, where we had three disjunctive terms in the mixed universal–existential case studied there. In the above QFF, (100) or (101), the symbol $r_{4,2}$ denotes the second real root of the cubic polynomial

$$p_4(s) := (s^3 - s^2 - s + 1)\,a_n b_n c_n - V_c = (s-1)^2(s+1)V_n - V_c. \tag{102}$$

With respect to the QFF (100) and, equivalently, (101) we observe that we have only one function $\widehat{\delta}$ related to the present info-gap model described in Subsection 5.1 and the related horizon of uncertainty h contrary to the previous subsection, Subsection 5.4, where we had two functions, the functions $\widehat{\gamma}_1 = \widehat{\gamma}_1(a_n, b_n, c_n, V_c)$ and $\widehat{\gamma}_2 = \widehat{\gamma}_2(a_n, b_n, c_n, V_c)$. The present function $\widehat{\delta}$ is the function

$$\widehat{\delta} = \widehat{\delta}(a_n, b_n, c_n, V_c) := r_{4,2} \tag{103}$$

with the symbol $r_{4,2}$ already defined. Using this notation, we can rewrite the QFF (101) in the form

$$V_n \le V_c \vee (V_n > V_c \wedge h \ge \widehat{\delta}). \tag{104}$$

Naturally, completely similar are the two cases where the universally quantified variable is the dimension b or the dimension c of the present rectangular cuboid instead of the dimension a of the same cuboid assumed and studied in this subsection. Therefore, we will pay no attention to these two completely analogous cases. The resulting QFF is again the QFF (100) or, equivalently, (101).

Here our conclusion is that whereas in the purely universal case having been studied in Subsection 5.2 we have one function related to the uncertainty parameter (horizon of uncertainty) h, i.e. the extremely well-known robustness function (or simply robustness) $\widehat{\alpha} = \widehat{\alpha}(a_n, b_n, c_n, V_c)$ and in the purely existential case having been studied in Subsection 5.3 we again have one function, the well-known opportuneness (or opportunity) function (or simply opportuneness) $\widehat{\beta} = \widehat{\beta}(a_n, b_n, c_n, V_c)$, on the contrary, in the two mixed universal–existential cases, we may have either one function, such as the function $\widehat{\delta} = \widehat{\delta}(a_n, b_n, c_n, V_c)$ in the present subsection, or two functions, such as the functions $\widehat{\gamma}_1 = \widehat{\gamma}_1(a_n, b_n, c_n, V_c)$ and $\widehat{\gamma}_2 = \widehat{\gamma}_2(a_n, b_n, c_n, V_c)$, as has been the case in the previous Subsection 5.4.

It is further understood that in the (mixed) universal–existential case, the case that is the topic of interest in this technical report, we may have more than two functions in the resulting QFFs (particularly if the number of uncertain variables exceeds two) although we will not intend to study such complicated cases here. Additionally, in the present (mixed) universal–existential case, we can also consider the case of more than one horizon of uncertainty as we already did in Subsection 3.5.

6. Reactions at the ends of a fixed (clamped) ordinary beam loaded by a concentrated load

6.1. The system model, the info-gap model, the performance requirements and the assumptions

As a final application of the present approach based on (i) Ben-Haim's IGDT (info-gap or information-gap decision theory) and (ii) quantifier elimination in this section we consider the problem of a fixed (clamped) ordinary beam (fixed at both its ends). The length of the beam is L (obviously with $L > 0$ and also with $0 \le x \le L$) and the beam is loaded by a concentrated normal load P (assumed to be a positive quantity, $P > 0$) applied to the interior point $x = a$ of the beam evidently with $0 < a < L$. This is a classical problem in structural mechanics and mechanics of materials (or strength of materials). The reactions R_A and R_B at the two ends of the beam A (left end with $x = 0$) and B (right end with $x = L$), respectively, are given by the formulae [114, Table 8.1, p. 190, case 1d]

$$R_A = \frac{P}{L^3}(L-a)^2(L+2a), \quad R_B = \frac{Pa^2}{L^3}(3L-2a). \tag{105}$$

These equations constitute the system model $\mathscr{M}_s$ in the present beam problem. Hence, this model has the form

$$\mathscr{M}_s := \left[R_A = \frac{P}{L^3}(L-a)^2(L+2a)\right] \wedge \left[R_B = \frac{Pa^2}{L^3}(3L-2a)\right]. \tag{106}$$

Of course, we should have and really have $R_A + R_B = P$. This fact can be directly verified manually or, alternatively, with the help of *Mathematica* [68].

Here we assume that the three parameters L, a and P in the present beam problem are uncertain variables. Next, the assumed info-gap model of uncertainty $\mathscr{U}_s$ in the same beam problem is again a fractional-error model. The final form of this model that is appropriate for quantifier elimination is

$$\mathscr{U}_s := (1-h)L_n \le L \le (1+h)L_n \wedge (1-h)a_n \le a \le (1+h)a_n \wedge (1-h)P_n \le P \le (1+h)P_n. \tag{107}$$

Here the symbol h again denotes the uncertainty parameter (or horizon of uncertainty). Additionally, this parameter is assumed to be the same for all three uncertain variables L, a and P. Moreover, the three symbols L_n, a_n and P_n denote the nominal values of these three uncertain variables L, a and P, respectively. Here we also assume that $0 < h < 1$. In this way, h is non-negative (as it should be) and, additionally, all three uncertain variables L, a and P are always positive quantities. This is clear from the info-gap model $\mathscr{U}_s$, which has been defined in Eq. (107).

Now, as far as the performance requirements $\mathscr{C}_s$ are concerned, here we assume that we have two such requirements. More explicitly, we wish that both reactions R_A and R_B at the fixed ends of the beam A (with $x = 0$) and B (with $x = L$), respectively, do not exceed a critical value R_c, which is the same for both of these ends A and B. Hence, here we have the two performance requirements

$$\mathscr{C}_s := R_A \le R_c \wedge R_B \le R_c, \tag{108}$$

where the two reactions R_A and R_B are given by Eqs. (105) or, alternatively, by the system model $\mathscr{M}_s$, which is determined by Eq. (106).

Finally, we also make the following assumptions:

$$\mathscr{A}_s := L > 0 \wedge 0 < a < L \wedge P > 0 \wedge L_n > 0 \wedge a_n > 0 \wedge P_n > 0 \wedge R_c > 0 \wedge 0 < h < 1. \tag{109}$$

These assumptions are mainly positivity assumptions, but they also concern the physical assumption $0 < a < L$ (the point $x = a$ is an interior point of the beam) and the aforementioned additional assumption $0 < h < 1$. The latter assumption, $0 < h < 1$, assures the positivity of the three uncertain variables L, a and P in the info-gap model of uncertainty $\mathscr{U}_s$, which has been defined in Eq. (107).

6.2. Quantifier elimination in the (mixed) universal–existential case with one parameter

In this uncertainty problem, we restrict our attention exclusively to the (mixed) universal–existential case. In this case, we wish that there exists at least one concentrated normal load P applied to the point $x = a$ of the beam such that the two performance requirements $\mathscr{C}_s$ defined in Eq. (108) hold true for all pairs of values (L, a) of the two uncertain variables L (length of the beam) and a (point of application of the load P). Therefore, here the following quantified formula holds true:

$\forall(L,a)\,\exists P$ in the info-gap model $\mathscr{U}_s$ such that if the system model $\mathscr{M}_s$ and the assumptions $\mathscr{A}_s$ hold true, then the two performance requirements $\mathscr{C}_s$ also hold true. (110)

The present problem is a rather difficult problem for quantifier elimination because it concerns three quantified variables, the uncertain variables L, a and P, and, additionally, five free variables, the four parameters L_n, a_n, P_n and R_c and the unknown horizon of uncertainty h. Therefore, eight variables (both free and quantified) are present. This fact makes the application of quantifier elimination (here with the help of *Mathematica*) to the above quantified formula (110) a very difficult and, most probably, an impossible task. Obviously, this situation is due to the doubly-exponential computational complexity of quantifier elimination proved by Davenport and Heintz in 1988 [69].

For this reason here we intend to simplify the above universally–existentially quantified formula (110) at first by assigning numerical values to the three nominal values L_n, a_n and P_n of the three uncertain variables L, a and P, respectively. Here we assume the following numerical values:

$$L_n = 10, \quad a_n = 3, \quad P_n = 100 \tag{111}$$

with the appropriate physical units, e.g. with meters (m) for L_n and a_n and kilonewtons (kN) for P_n. Therefore, now only two free variables are still present: the critical value R_c in the performance requirements $\mathscr{C}_s$ in Eq. (108) and the horizon of uncertainty h in the info-gap model $\mathscr{U}_s$ in Eq. (107).

Now we are ready to proceed adopting the following way of thinking: We take into account that (i) both reactions R_A and R_B in the system model $\mathscr{M}_s$ in Eq. (106) are proportional to the normal load P applied to the point $x = a$ of the beam, (ii) in the performance requirements $\mathscr{C}_s$ in Eq. (108), we wish that the same reactions R_A and R_B do not exceed a critical value R_c and (iii) in the quantified formula (110), we wish the existence of at least one value of the normal load P for the satisfaction of the performance requirements $\mathscr{C}_s$. Then it is evident that this value of the normal load P can be selected by us in advance simply as the least possible value P_min of this load. Now, according to the info-gap model of uncertainty $\mathscr{U}_s$ in Eq. (107) (third conjunctive term there) this least possible value is simply $P_\mathrm{min} = (1-h)P_n$ and we intend to use it instead of the normal load P in the quantified formula and, next, during quantifier elimination. Thus the number of quantified variables reduces to two, the variables L and a, both of which are now universally quantified in the initial quantified formula (110). Under these conditions this quantified formula now takes the (purely) universal form

$\forall(L,a)$ in the info-gap model $\mathscr{U}_s$ and $P = P_\mathrm{min} = (1-h)P_n$ if the system model $\mathscr{M}_s$ and the assumptions $\mathscr{A}_s$ hold true, then the two performance requirements $\mathscr{C}_s$ also hold true (112)

with $P_n = 100$ as has been already assumed in the last of Eqs. (111) and now only two quantified variables (L and a) and two free variables (R_c and h). In this way, it is expected that *Mathematica* can now successfully perform quantifier elimination to this formula and, actually, this is the case.

For this quantifier elimination with $P = P_\mathrm{min} = (1-h)100$ we can use the related command

```
Refine[Reduce[ForAll[{L,a}, Us[[{1,2}]] ∧ As, Cs], Reals], As]//Factor        [c30]
```

The resulting QFF has the following logical form consisting of three disjunctive terms (three cases for the horizon of uncertainty h):

$$\begin{aligned}&\Big[h \le \frac{1}{2} \wedge R_\mathrm{c} \ge \frac{2(1-h)(4+h)(7+13h)^2}{5(1+h)^3}\Big]\\ \vee\ &\Big[\frac{1}{2} < h \le \frac{7}{13} \wedge R_\mathrm{c} \ge \frac{54(1+h)^2(2-3h)}{5(1-h)^2}\Big]\\ \vee\ &\Big[\frac{7}{13} < h \wedge R_\mathrm{c} \ge 100(1-h)\Big].\end{aligned} \tag{113}$$

It is clear that the above QFF (113) is appropriate mainly for the determination of the critical value R_c of both reactions R_A and R_B at the ends A (with $x = 0$) and B (with $x = L$), respectively, of the present fixed (clamped) ordinary beam in the case where the uncertainty parameter (or horizon of uncertainty) h is known in advance. But here we are particularly interested in the inverse problem and this is the problem of determination of the horizon of uncertainty h when the critical value R_c is known in advance even as a parameter as is here the case. For this reason the above QFF (113), which was automatically (i.e. without any related instruction in the quantifier elimination command) solved with respect to R_c, is inappropriate for our task and it should appear solved with respect to h. To this end it is preferable to use the slightly modified quantifier elimination command

```
Refine[Reduce[ForAll[{L,a},Us[[{1,2}]] ∧ As,Cs],h,Reals],As]                [c31]
```

with the use of the additional argument `h` in the `Reduce` command used for quantifier elimination.

The resulting QFF now has the logically equivalent but completely different in appearance form

$$\begin{aligned}
&\left\{R_c \leq \frac{600}{13} \wedge h \geq 1 - \frac{R_c}{100}\right\}\\
\vee &\left\{\frac{600}{13} < R_c \leq \frac{243}{5}\right.\\
&\quad \left.\wedge\, h \geq \mathrm{Root}[162s^3 + (5R_c + 216)s^2 - (10R_c + 54)s + 5R_c - 108, 3]\right\}\\
\vee &\left\{\frac{243}{5} < R_c < \frac{392}{5}\right.\\
&\quad \left.\wedge\, h \geq \mathrm{Root}[338s^4 + (5R_c + 1378)s^3 + (15R_c - 162)s^2 + (15R_c - 1162)s + 5R_c - 392, 4]\right\}\\
\vee\ & R_c \geq \frac{392}{5}
\end{aligned} \tag{114}$$

because now this QFF is solved with respect to the horizon of uncertainty h as was here our wish. In the above QFF (114), the symbol $\mathrm{Root}[p(s), n]$ denotes the nth real root of the polynomial $p(s)$. We observe that the above QFF (114) consists of four disjunctive terms referring to the values of the parameter R_c, which is the critical value of both reactions R_A and R_B at the ends A (with $x = 0$) and B (with $x = L$), respectively, of the present fixed beam. Now by using the `N` command of *Mathematica* for numerical approximations we can rewrite the above QFF (114) in the equivalent decimal form

$$\begin{aligned}
&\left\{R_c \leq 46.1538 \wedge h \geq 1 - 0.01R_c\right\}\\
\vee &\left\{46.1538 < R_c \leq 48.6\right.\\
&\quad \left.\wedge\, h \geq \mathrm{Root}[162s^3 + (5R_c + 216)s^2 - (10R_c + 54)s + 5R_c - 108, 3]\right\}\\
\vee &\left\{48.6 < R_c < 78.4\right.\\
&\quad \left.\wedge\, h \geq \mathrm{Root}[338s^4 + (5R_c + 1378)s^3 + (15R_c - 162)s^2 + (15R_c - 1162)s + 5R_c - 392, 4]\right\}\\
\vee\ & R_c \geq 78.4.
\end{aligned} \tag{115}$$

From the QFF (114) we observe that in the present (mixed) universal–existential case, which concerns the quantified formula (110) together with the numerical nominal values (111) of the three uncertain variables (or uncertain parameters) L, a and P (although for computational convenience we have used the equivalent form (112) of the quantified formula (110) during quantifier elimination) the uncertainty parameter (or horizon of uncertainty) h, which is of particular interest here, is determined by an inequality of the form

$$h \geq \widehat{\varepsilon}(R_c). \tag{116}$$

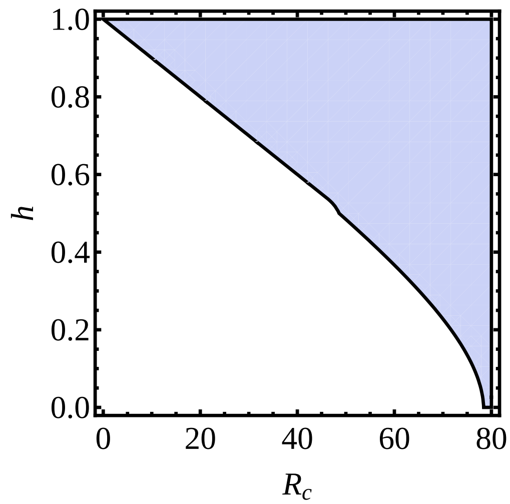

Fig. 1. The reliability region $\mathscr{R}_s$ that corresponds to the QFF (113) or to the logically equivalent QFF (114) or (115) for the nominal values $L_n = 10$, $a_n = 3$ and $P_n = 100$ in Eqs. (111) of the three uncertain variables L, a and P in the present beam problem under a concentrated normal load P.

Now it is clear that because of the QFF (114) the new function $\widehat{\varepsilon}(R_c)$ (the greatest lower bound or infimum, inf, of the horizon of uncertainty h, here also its minimum) is determined by

$$\widehat{\varepsilon}(R_c) = \begin{cases} 1 - \dfrac{R_c}{100} & \text{if } R_c \leq \dfrac{600}{13}, \\ r_{5,3} & \text{if } \dfrac{600}{13} < R_c \leq \dfrac{243}{5}, \\ r_{6,4} & \text{if } \dfrac{243}{5} < R_c < \dfrac{392}{5}, \\ 0 & \text{if } R_c \geq \dfrac{392}{5}. \end{cases} \tag{117}$$

In this equation, the new symbol $r_{5,3}$ denotes the third real root of the parametric cubic polynomial

$$p_5(s) := 162s^3 + (5R_c + 216)s^2 - (10R_c + 54)s + 5R_c - 108 \tag{118}$$

whereas the new symbol $r_{6,4}$ denotes the fourth real root of the parametric quartic polynomial

$$p_6(s) := 338s^4 + (5R_c + 1378)s^3 + (15R_c - 162)s^2 + (15R_c - 1162)s + 5R_c - 392. \tag{119}$$

Both of these polynomials, $p_5(s)$ and $p_6(s)$, have the critical value R_c of the reactions R_A and R_B at the two ends A and B, respectively, of the present fixed ordinary beam as a parameter in almost all of their coefficients (more explicitly, in all of their coefficients with the exception of the first one).

In Fig. 1, we display the reliability region $\mathscr{R}_s$ that corresponds to the QFF (113) or, alternatively, to the logically equivalent QFF (114) or (115) of course for the three nominal values $L_n = 10$, $a_n = 3$ and $P_n = 100$ in Eqs. (111) of the uncertain variables (parameters) L, a and P in the present fixed beam problem under a concentrated normal load P. From this figure, Fig. 1, we directly observe that the greatest lower bound $\widehat{\varepsilon}(R_c)$ of the uncertainty parameter (or horizon of uncertainty) h decreases for increasing values of the critical value R_c. This fact will also be verified in the next subsection.

6.3. Quantifier elimination in the (mixed) universal–existential case without a parameter

The whole situation with respect to quantifier elimination becomes much easier in the simpler case where we assign a numerical value to the critical value R_c of the reactions R_A and R_B in the performance requirements $\mathscr{C}_s$ defined in Eq. (108). For example, we may assume that $R_c = 1$. Again we also assume the validity of the numerical values $L_n = 10$, $a_n = 3$ and $P_n = 100$ in Eqs. (111) for the nominal values L_n, a_n and P_n, respectively. Under these conditions in the present (mixed) universal–existential (AE or $\forall\exists$) case, we can directly employ the quantified formula in its initial form (110) instead of its computationally simpler (although here essentially equivalent) modified form (112). The latter form is free from the existential quantifier $\exists$ and it was already successfully used in the previous subsection for the derivation of the related QFFs (quantifier-free formulae).

In the present simpler case, we can use the initial, unsimplified quantifier elimination command

```
Refine[Reduce[ForAll[{L,a},Us[[{1,2}]] ∧ As,Exists[P,Us[[3]] ∧ As,Cs]],
        Reals],As]                                                    [c32]
```

which now includes both quantifiers $\forall$ (`ForAll`) and $\exists$ (`Exists`). The three quantified variables to be eliminated are still L, a and P, but now only the horizon of uncertainty h remains a free variable.

The resulting QFFs for several numerical values of the critical value R_c in the performance requirements $\mathscr{C}_s$ defined in Eq. (108) have the following and now really sufficiently simple forms:

$$h \geq \frac{99}{100} = 0.99 \qquad \text{for } R_c = 1, \tag{120}$$

$$h \geq \frac{9}{10} = 0.90 \qquad \text{for } R_c = 10, \tag{121}$$

$$h \geq \frac{4}{5} = 0.80 \qquad \text{for } R_c = 20, \tag{122}$$

$$h \geq \frac{7}{10} = 0.70 \qquad \text{for } R_c = 30, \tag{123}$$

$$h \geq \frac{3}{5} = 0.60 \qquad \text{for } R_c = 40, \tag{124}$$

$$h \geq \mathrm{Root}[169s^4 + 814s^3 + 294s^2 - 206s - 71, 4] \approx 0.484367 \qquad \text{for } R_c = 50, \tag{125}$$

$$h \geq \mathrm{Root}[169s^4 + 839s^3 + 369s^2 - 131s - 46, 4] \approx 0.366684 \qquad \text{for } R_c = 60, \tag{126}$$

$$h \geq \mathrm{Root}[169s^4 + 864s^3 + 444s^2 - 56s - 21, 4] \approx 0.227965 \qquad \text{for } R_c = 70, \tag{127}$$

$$h \geq \mathrm{Root}[338s^4 + 1769s^3 + 1011s^2 + 11s - 1, 4] \approx 0.0259898 \quad \text{for } R_c = \frac{391}{5} = 78.2, \tag{128}$$

$$h \geq \mathrm{Root}[3380s^4 + 17699s^3 + 10137s^2 + 137s - 1, 4] \approx 0.00524504 \quad \text{for } R_c = \frac{3919}{50} = 78.38, \tag{129}$$

$$h > 0 \quad \text{(any positive number; result (QFF): } \texttt{True}) \qquad \text{for } R_c = \frac{392}{5} = 78.4, \tag{130}$$

$$h > 0 \quad \text{(any positive number; result (QFF): } \texttt{True}) \qquad \text{for } R_c = 80, 90 \text{ and } 100. \tag{131}$$

From the above QFFs we observe that the greatest lower bound (infimum, inf, here also minimum) $\widehat{\varepsilon} = \widehat{\varepsilon}(R_c)$ of the horizon of uncertainty h decreases as the critical value R_c increases and this is completely reasonable. More explicitly, with increasing values of R_c the satisfaction of the performance requirements $\mathscr{C}_s$ defined in Eq. (108) becomes easier and, hence, possible for lower values of h because in the present uncertainty problem (of course, not always) we must have $h \geq \widehat{\varepsilon}(R_c)$ according to the inequality (116). Additionally, when $R_c \geq 392/5 = 78.4$, there is no restriction at all on the horizon of uncertainty h beyond the assumption that it is a positive number. This is also clear from the QFF (114) and, numerically, (115) or from the function $\widehat{\varepsilon}(R_c)$ determined by Eq. (117). It has also been verified that the above QFFs (120)–(131) can also be obtained as special cases of the general QFF (114) and, numerically, (115) (with parameter R_c) for the corresponding values of R_c.

7. Conclusions–discussion

From all the above results it is concluded that the computational method of quantifier elimination for real variables implemented in some computer algebra systems (*Reduce*, *Mathematica* and *Maple*) constitutes an interesting computational tool in Ben-Haim's IGDT (info-gap or information-gap decision theory) not only with respect to the robustness and the opportuneness (or opportunity) functions related to (purely) universally and (purely) existentially quantified formulae, respectively, as is already known, but also to the general case of mixed quantified formulae with the presence of both the universal and the existential quantifiers for the related uncertain variables. Of course, the presence of both the universal and the existential quantifiers in a quantified formula concerns only the case of more than one uncertain variable in the problem under consideration by using the IGDT.

The applications here concerned only (i) two geometry problems (the area of a rectangle and the volume of a rectangular cuboid) and (ii) two applied mechanics problems (the buckling load of a fixed–free column and the reactions at the ends of a fixed ordinary beam loaded by a concentrated load) in all cases under uncertainty conditions and by using Ben-Haim's IGDT. Nevertheless, it is clear that the present approach is also applicable to any problem under uncertainty conditions studied by this interesting approach, Ben-Haim's IGDT, clearly also in the (mixed) case of universally–existentially quantified formulae. This class of quantified formulae constitutes a new case of use of the IGDT that is studied here by using the method of quantifier elimination beyond the previously studied (purely) universally and (purely) existentially quantified formulae again by using the IGDT.

It is well known that the (purely) universal case and the related robustness function is the case where we wish that the performance requirements hold true for any value(s) of the uncertain variable(s) in the problem under consideration described by the related system model and satisfying the assumed info-gap model. Quite similarly, the (purely) existential case and the related opportuneness (or opportunity) function is the case where we wish that the performance requirements hold true for at least one value (one set of values) of the uncertain variable(s) in the problem under consideration again described by the related system model and satisfying the assumed info-gap model. Naturally, both these fundamental cases are of particular importance in the related problems, which are both of practical importance under the related conditions in the physical problem that is studied.

Here the case of interest is the third, the mixed case. In this case, we wish that the performance requirements hold true (i) for all values (universal quantifier $\forall$, for all) of one (or more than one) particular uncertain variable, but also (ii) for at least one value (existential quantifier $\exists$, exists) of one (or more than one) clearly different particular uncertain variable of course under the validity of the system model and the assumed info-gap model. In this author's opinion, this intermediate case, the mixed case, may also be of interest in several practical problems especially in design problems and, therefore, it may deserve some further attention inside the powerful and popular Ben-Haim's IGDT.

Evidently, as was already observed, in the present mixed case, additional functions appear in the resulting QFFs (quantifier-free formulae) beyond the extremely well-known and popular robustness and opportuneness (or opportunity) functions having been studied in detail so far. From the QFFs derived in the previous sections it is directly observed that although the robustness functions always concern least upper bounds (suprema, sup) of the horizons of uncertainty and the opportuneness (or opportunity) functions always concern greatest lower bounds (infima, inf) of the horizons of uncertainty, here, in the mixed case, the new related functions may concern either (i) least upper bounds or (ii) greatest lower bounds or even (iii) both least upper bounds and greatest lower bounds in the same QFF (QFF (96) in Subsection 5.4) with the appearance of two functions inside the same QFF.

Finally, here it is repeated that, unfortunately, the method of quantifier elimination has a doubly-exponential computational complexity as was proved by Davenport and Heintz in 1988 [69]. Hence, the use of this approach is applicable only to problems with a small total number of variables (both quantified and free). Usually, in most cases (but not always), this number should not exceed 5 to 7.

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