Computer-generated formulae
for the location of straight cracks

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TECHNICAL NOTE

COMPUTER-GENERATED FORMULAE
FOR THE LOCATION OF STRAIGHT CRACKS

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Abstract—Complex path-independent integrals have been widely used for the location of cracks, holes and inclusions in plane isotropic elasticity problems. Here a much simpler and more direct approach, applicable to particular inverse straight crack problems in infinite elastic media, is suggested. This approach is based on the use of experimental data for the derivative of the first complex potential of Kolosov–Muskhelishvili from few points only of the elastic specimen (not along a whole closed contour) and, next, on the construction of appropriate polynomial equations for the determination of the position of the crack tips, the stress-intensity factors, etc. These equations can be derived by using computer algebra methods (such as Gröbner bases and characteristic sets) and the related commercially available software (such as Maple V). The present elementary approach is illustrated in two simple cases of straight cracks and the related formulae are displayed. Further possibilities are also suggested in brief.

INTRODUCTION

METHODS based on complex path-independent integrals have been widely used for the location of zeros and poles of analytic/meromorphic functions in complex analysis (see e.g. the review paper [1]). On the other hand, similar methods were suggested for plane elasticity problems and, particularly, crack problems [2] and received sufficient attention in the literature (see e.g. [3]). This approach was further generalized from the location of zeros and poles of analytic/meromorphic functions [1] to the location of cracks [4] as well as holes and inclusions [5, 6]. The cases of interface cracks [7] as well as curvilinear cracks [8] have also been considered by the same method. Beyond classical plane elasticity [2–8], complex path-independent integrals have been also applied to elastic plate problems and to the related location of circular holes in plates [9].

In spite of the interest and the powerfulness of the method of complex path-independent integrals in elasticity problems and, mainly, in the location of cracks, holes and inclusions, a possible disadvantage of this method is that experimental data are required to be gathered along a whole closed contour $C$ on the elastic plane. Here we will suggest an alternative method for the location of cracks in classical plane isotropic elasticity, which is much more direct and easy to use, since it is based on available experimental data obtained only at a small number of points $P_k$ on the plane equal to the number $n$ of the unknown quantities to be determined (such as the positions of the crack tips $a$ and $b$ and the stress-intensity factors $K$ at these tips) possibly plus one or two. Moreover, in the present method, numerical integration along a complex contour $C$ (as is the case in complex path-independent integrals) is completely avoided. The present approach is valid only when the number of these unknown quantities is small and assumes that we have available the theoretical equation for the derivative $\Phi(z)$ of the first complex potential $\phi(z)$ of Kolosov–Muskhelishvili [10] including, of course, the unknown quantities to be determined.
The present approach is based on the solution of appropriate systems of \( n \) polynomial equations, which can be made by standard computer algebra algorithms (such as the classical Buchberger algorithm for the construction of Gröbner bases [11–13] or, almost equivalently, the related algorithm for the construction of characteristic sets [13, 14]). Both of these algorithms are available in Maple V [15] (the second one is standard only since release 2, 1993) and the first of them is also standard in Axiom, Macsyma, Mathematica and Reduce. Here we will use the Buchberger’s algorithm implementation in Maple V [15, pp. 469–478]. In passing, we can add that Gröbner bases, beyond their classical applications in applied mathematics, geometry and inverse robot kinematics [11, 16], were recently used also in elasticity problems [17] as well as in special fracture mechanics problems [18, 19].

THE APPROACH

We consider an infinite plane isotropic elastic medium with a single straight crack \( L = [a, b] \), the position of which is not known to us. We wish to determine the positions of the crack tips \( z = a \) and \( z = b \) as well as the related loading and additional parameters. To this end we assume that we have gathered experimental data from a few number of points \( P_k (k = 1,2,\ldots,n) \) of the elastic medium, with complex coordinates \( z_k = x_k + iy_k \), by using the classical methods of experimental stress analysis [20] and the more sophisticated methods described in the specialized related journals such as Experimental Mechanics and Experimental Techniques. By using these data, we can compute the values \( \Phi_k \) of the derivative \( \Phi(z) \) of the first complex potential \( \phi(z) \) of Kolosov–Muskhelishvili [10] at the points \( P_k \).

Next, by assuming an appropriate form for \( \Phi(z) \), e.g.

\[
\Phi(z) = \frac{C_0 z + C_1}{\sqrt{(z-a)(z-b)}}, \quad z = x + iy, \tag{1}
\]

which holds true for a single straight unloaded crack \( L \) inside an infinite plane isotropic elastic medium under a constant loading at infinity [10], we observe that we have to determine \( n = 4 \) unknown quantities: the crack tips \( z = a \) and \( z = b \) as well as the constants \( C_0 \) and \( C_1 \). Frequently, this number \( n \) of unknowns may be even smaller: for example, we may know the intensity \( p_0 \) of loading at infinity whence \( C_0 \) is known in advance. Similarly, an assumed symmetry in the crack position about the centre \( O = (0,0) \) of the selected \( Ox \)-Cartesian coordinate system (with the position of the point \( O \) and the directions of the axes \( Ox \) and \( Oy \) known in advance) leads to \( b = -a \) and so on. In any case, if we have \( n \) unknown quantities in (1), we need at least \( n \) points \( P_k \) with experimental data for \( \Phi(z) \). Then, in principle, we are able to solve the generally nonlinear system of polynomial equations

\[
\Phi_k = \frac{C_0 z_k + C_1}{\sqrt{(z_k-a)(z_k-b)}}, \quad k = 1,2,\ldots,n, \tag{2}
\]

and, therefore, determine the aforementioned unknown quantities.

Of course, beyond the location of the crack \( L \) (through the determination of its tips \( z = a \) and \( z = b \)), we can also determine the stress-intensity factors \( K_{a,b} \) at the crack tips, since

\[
K_a = 2\sqrt{2\pi} \lim_{z \to a} \sqrt{z-a} \Phi(z) \tag{3}
\]

and a similar formula holds true for the other crack tip \( z = a \) too.

The solution of the \( n \) equations of the system of polynomial equations (2) for the unknown quantities in (1) can be achieved by using popular computer algebra methods already having been mentioned in the previous section. In the applications below, we will use Maple V and the related \texttt{gbasis} command in the \texttt{gröbner} package for the construction of Gröbner bases [15, pp. 469–478].
Finally, as far as the experimental determination of $\Phi(z)$ is concerned, we can use the classical formulae [10, pp. 114, 118]
\[
\sigma_y - i\sigma_{xy} = \Phi(z) + k\Phi(z) + z\Phi'(z) + \Psi(z), \tag{4}
\]
\[
2\mu(u + iv) = k\phi(z) - z\Phi(z) - \psi(z), \tag{5}
\]
where $\sigma_x$ and $\sigma_{xy}$ are stress components, $\mu$ is the shear modulus, $k$ is the Muskhelishvili constant [10, p. 112] and $\phi(z)$ and $\psi(z)$ are the classical complex potentials of Kolosov–Muskhelishvili [10, pp. 109–112] (the first derivatives of $\phi(z)$ and $\psi(z)$ denoted by $\Phi(z)$ and $\Psi(z)$, respectively [10, p. 114]).

By differentiating (5) with respect to $x$ and, next, adding the resulting equation to (4), we find
\[
\Phi(z) = \left[ (\sigma_y - i\sigma_{xy}) + 2\mu \left( \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right) \right] / (\kappa + 1). \tag{6}
\]

By using this equation and the available experimental data at the points $P_k$, we can directly determine the required quantities $\Phi_k$ at the left-hand side of (2).

A somewhat more general equation than (5) is reported by Tsamasphyros and Theocaris [3]:
\[
\Phi(t) = \left[ (\sigma_n - i\sigma_t) + 2\mu \left( \frac{\partial u}{\partial t} + i\frac{\partial v}{\partial t} \right) \right] / (\kappa + 1), \tag{7}
\]
where $t = x + iy$ denotes the complex coordinate $z$ of the points of a smooth contour $C$ including the point $P$ under consideration and the subscripts $n$ and $t$ refer to the normal and tangential directions to $C$ at this point $t$, respectively. (Obviously, if $C$ coincides with the straight line through $P$ parallel to the $Ox$-axis, then (7) reduces to (6).)

**APPLICATIONS**

Now we will illustrate the above approach in two simple applications by using special cases of (1).

*First application:* We assume that we know the constants $C_0$ and $C_1$ in $\Phi(z)$ in (1) and that we wish to determine only the positions of the crack tips $a$ and $b$. (Of course, $a$ and $b$ may be not only real but also complex numbers in the elastic plane.) To this end, we require two points $P_k$ for the gathering of the experimental data $\Phi_k$ described in detail in the previous section.

By using these two points, $P_1$ and $P_2$, we easily get from (2) the following system of two polynomial equations in two unknowns: $a$ and $b$
\[
(z_k - a)(z_k - b)\Phi_k^2 - (C_0z_k + C_1)^2 = 0, \quad k = 1, 2. \tag{8}
\]

By solving these equations with the Gröbner basis method [15, pp. 469–478], we easily find the following quadratic equation for $b$
\[
(z_1 - z_2)^2\Phi_1^2\Phi_2^2b^2 + [2C_0C_1(z_1\Phi_2^2 - z_2\Phi_1^2) + C_0^2(z_1^2\Phi_2^2 - z_2^2\Phi_1^2) - C_1^2(\Phi_1^2 - \Phi_2^2)] - (z_1^2 - z_2^2)\Phi_1^2\Phi_2^2b + 2C_0C_1z_1z_2(\Phi_1^2 - \Phi_2^2) - C_0^2z_1z_2(z_1\Phi_2^2 - z_2\Phi_1^2) + C_1^2(z_1\Phi_1^2 - z_2\Phi_2^2) + z_1z_2(z_1 - z_2)\Phi_1^2\Phi_2^2 = 0. \tag{9}
\]

This equation possesses two roots, but its symmetry (combined with the symmetry in (1) with respect to $a$ and $b$) reveals us that these roots correspond to both $a$ and $b$. This can also be verified by giving a priority to $a$ (over $b$) in the lexicographic ordering and reconstructing the above Gröbner basis. Then we observe that $a$ satisfies also (9).
After the determination of the crack tips \(a\) and \(b\) from (9), we can easily check our results by using some other element of the present Gröbner basis, e.g. the obvious equation

\[
(z_1 - a)(z_1 - b)\Phi_1^2 - (C_0 z_1 + C_1)^2 = 0
\]  
\(\text{(10)}\)

or the corresponding equation for the point \(P_2\), both of which should be verified. Therefore, we have been able to determine both crack tips \(a\) and \(b\) in the present application.

**Second application:** We assume now that \(C_1\) in (1) vanishes and that we have three unknown quantities: \(C_0\), \(a\) and \(b\) (instead of two in the previous application). In this case, we must use three points \(P_k\) for the experimental measurement of \(\Phi_k\) and the polynomial equations (8) take the form

\[
(z_k - a)(z_k - b)\Phi_k^2 - (C_0 z_k)^2 = 0, \quad k = 1, 2, 3.
\]  
\(\text{(11)}\)

By computing the resulting Gröbner basis (exactly as previously), we found again a quadratic equation for the crack tip \(b\) that is

\[
\left[(z_1 - z_2)z_3^2\Phi_1^2\Phi_3^2 + (z_2 - z_3)z_1^2\Phi_2^2\Phi_3^2 + (z_3 - z_1)z_2^2\Phi_3^2\Phi_1^2\right]b^2
- \left[(z_1^2 - z_2^2)z_3^2\Phi_1^2\Phi_3^2 + (z_1^2 - z_3^2)z_2^2\Phi_2^2\Phi_3^2 + (z_2^2 - z_3^2)z_1^2\Phi_3^2\Phi_1^2\right]b
+ z_1z_2z_3[z_3(z_1 - z_2)\Phi_1^2\Phi_2^2 + z_1(z_2 - z_3)\Phi_2^2\Phi_3^2 + z_2(z_3 - z_1)\Phi_3^2\Phi_1^2 - \Phi_1^2\Phi_3^2\Phi_2^2 - \Phi_1^2\Phi_2^2\Phi_3^2 - \Phi_2^2\Phi_1^2\Phi_3^2] = 0.
\]  
\(\text{(12)}\)

Exactly as in the previous application, (12) has two roots and the symmetry of the problem and (12) itself reveal that these roots are the sought complex coordinates \(a\) and \(b\) of the crack tips.

Furthermore, we can check these roots by using one more element of the derived Gröbner basis (exactly as previously), e.g.

\[
[(z_2^2\Phi_1^2 - z_1^2\Phi_2^2)b + z_1z_2(z_1\Phi_2^2 - z_2\Phi_1^2)]a + z_1z_2(z_1\Phi_2^2 - z_2\Phi_1^2)b + z_1^2z_2^2(\Phi_1^2 - \Phi_2^2) = 0.
\]  
\(\text{(13)}\)

Finally, as far as the computation of \(C_0\) is concerned, if it is required, it can be made from a third element of the same Gröbner basis, e.g. (on the basis of the point \(P_1\))

\[
z_1^2C_0^2 - [z_1^2 - (a + b)z_1 + ab]\Phi_1^2 = 0
\]  
\(\text{(14)}\)

or from a similar equation for one of the points \(P_2\) or \(P_3\).

Additional related crack problems were also studied by the present approach. For the sake of space, we will not proceed to display further related results, but we can notice that the resulting polynomial equations become much more complicated (and even more time-consuming inside the computer) when the number of unknown quantities in our fundamental formula for \(\Phi(z)\) increases.

**CONCLUSIONS—GENERALIZATIONS**

From the above results it is directly concluded that the old method of complex path-independent integrals for the location of cracks in plane elasticity problems can be substituted by a simpler and more direct method based on experimental data not along a whole closed contour, but only at an appropriate number of points generally equal to the number of unknown quantities (including the complex coordinates for the crack tips) in the assumed formula for the derivative \(\Phi(z)\) of the first complex potential \(\phi(z)\) of Kolosov–Muskhelishvili.

As was already observed, computer algebra algorithms constitute a convenient tool for the execution of the necessary computations and the derivation of symbolic formulae of general validity for the determination of the present unknown quantities. Of course, these quantities should be
limited in number; otherwise the required computer time will increase exponentially and the derived
formulae will be extremely complicated.

We can also add that, instead of \( \Phi(z) \), we can use its derivative \( \Phi'(z) \) in our computations
or any other appropriate complex quantity \( p(z) \), related to the Kolosov–Muskhelishvili complex
potentials \( \phi(z) \) and \( \psi(z) \), which can be measured at distinct points of the elastic specimen. In all
cases, provided that the assumed formula for \( p(z) \) is correct, beyond the determination of the crack
tips, the whole elasticity problem can be completely solved and the values of the stress-intensity
factors at the crack tips can be directly computed.

As a check of the final results by the present method, we can use one or two more points \( P_k \)
\((k = n + 1, n + 2)\) of the elastic medium, measure the values of \( \Phi_k \) at these points too and check
whether the resulting equations are verified (after the substitution of the numerical values for the
crack tips and the additional constants in the formula for \( \Phi(z) \)) or not. If they are, then we are sure
that our assumption for the formula for \( \Phi(z) \) is the correct one as is also the case for the obtained
numerical results. Otherwise (in the case of a wrong assumption), the whole procedure should be
repeated.

On the other hand, one additional point, \( P_{n+1} \), is also required when we have to determine
a single quantity \( q \) from a polynomial equation and we do not know in advance which root of
this equation is the correct one. Then we have to construct the same equation, but using also the
additional point \( P_{n+1} \), find its roots too and determine the common root of the above two polynomial
equations. This root is the correct value of \( q \).

Finally, the present approach for inverse elasticity problems (since the location of the crack
is not known in advance) although used here simply for cracks, can also be generalized to other
problems such as holes, ordinary inclusions, rigid line inclusions, etc. of course always under the
restriction that the closed-form formula for \( \Phi(z) \) is known in advance and it includes only a small
number of unknown quantities having to be experimentally determined in this formula. Similarly,
the present results can by generalized to anisotropic media, to simple plate problems, etc. The case
of simple arrays of cracks (such as a periodic array of collinear cracks) can also be considered by
the same approach, provided that new variables are used for the sine and cosine functions appearing
in the formula for \( \Phi(z) \) (in the case of a periodic array of cracks) and the fundamental trigonometric
identity \( \cos^2 z + \sin^2 z = 1 \) is also used during the construction of the appropriate Gröbner basis in
this application.

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